

A HILBERT MANIFOLD STRUCTURE ON THE REFINED TEICHMÜLLER SPACE OF BORDERED RIEMANN SURFACES

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ABSTRACT. We consider bordered Riemann surfaces which are biholomorphic to compact Riemann surfaces of genus g with n regions biholomorphic to the disc removed. We define a refined Teichmüller space of such Riemann surfaces and demonstrate that in the case that $2g + 2 - n > 0$, this refined Teichmüller space is a Hilbert manifold. The inclusion map from the refined Teichmüller space into the usual Teichmüller space (which is a Banach manifold) is holomorphic.

We also show that the rigged moduli space of Riemann surfaces with non-overlapping holomorphic maps, appearing in conformal field theory, is a complex Hilbert manifold. This result requires an analytic reformulation of the moduli space, by enlarging the set of non-overlapping mappings to a class of maps intermediate between analytically extendible maps and quasiconformally extendible maps. Finally we show that the rigged moduli space is the quotient of the refined Teichmüller space by a properly discontinuous group of biholomorphisms.

1. INTRODUCTION

In this paper, we construct a refined Teichmüller space of bordered Riemann surfaces of genus g with n boundary curves homeomorphic to the circle. If $2g + 2 - n > 0$ this refined Teichmüller space possesses a Hilbert manifold structure, and furthermore the inclusion map from this refined Teichmüller space into the standard one is holomorphic. In brief, the approach can be summarized as follows: we combine the results of Takhtajan and Teo [27] and Guo Hui [14] refining the universal Teichmüller space, with the results of Radnell and Schippers [23, 24, 25] demonstrating the relation between a moduli space in conformal field theory and the Teichmüller space of bordered surfaces. We also require a result by Nag [20, 21] on the variational method of Gardiner and Schiffer [10], together with the theory of marked holomorphic families of Riemann surfaces (see for example [6, 21, 17]). The demonstration that the transition functions of the atlas defining the Hilbert manifold structure are biholomorphisms, brings us into the realm of Besov spaces and the theory of Carleson measures for analytic Besov spaces. We also utilize the relationship between the Dirichlet space and the little Bloch space.

Our results are motivated both by Teichmüller theory, where there has been interest in refining Teichmüller space (see below), and by conformal field theory, where our results are required to solve certain analytic problems in the construction of conformal field theory from vertex operator algebras following Yi-Zhi Huang [15]. First, we give some background for the problem, and then outline our approach.

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There have been several refinements of quasiconformal Teichmüller space, obtained by considering natural analytic subclasses either of the quasisymmetries of the circle or of the quasiconformally extendible univalent functions in the Bers model of universal Teichmüller space. For example, Astala and Zinsmeister [3] give a model of the universal Teichmüller space based on BMO, and Cui and Zinsmeister [5] studied the Teichmüller spaces compatible with Fuchsian groups in this model. Gardiner and Sullivan [11] study a refined class of quasisymmetric mappings (which they call symmetric) and the topology of this refined class.

A family of refined models of the universal Teichmüller space was given by Guo Hui [14], each based on an L^p norm. These spaces were completely characterized in three ways: in terms of a space of quadratic differentials, in terms of univalent functions, and in terms of a space of Beltrami differentials; all satisfying a weighted L^p -type integrability condition. In this paper, we are concerned with the L^2 case. Guo Hui attributes the L^2 case to a preprint of Guizhen Cui, which we were unable to locate. Independently, Takhtajan and Teo [27] defined a Hilbert manifold structure on the universal Teichmüller space and universal Teichmüller curve, equivalent to that of Guo Hui, and obtained far-reaching results. These results include (among many others) obtaining a convergent Weil-Petersson metric and computation of its sectional curvatures, showing that the Kirillov-Yuri'ev-Nag-Sullivan period matrix is a holomorphic embedding of the universal Teichmüller space, and obtaining equivalent characterizations of elements of their refined universal Teichmüller space in terms of the generalized Grunsky matrix.

In conformal field theory one considers a moduli space originating with Friedan and Shenker [9]. We will use two different formulations of this moduli space due to Segal [26] and Vafa [29]. Vafa's *puncture model* of the rigged moduli space consists of equivalence classes of pairs (Σ, ϕ) , where Σ is a compact Riemann surface with n punctures, and $\phi = (\phi_1, \dots, \phi_n)$ is an n -tuple of one-to-one holomorphic maps from the unit disc $\mathbb{D} \subset \mathbb{C}$ into the Riemann surface with non-overlapping images. Two such pairs (Σ_1, ϕ) and (Σ_2, ψ) are equivalent if there is a biholomorphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ such that $\psi_i = \sigma \circ \phi_i$ for $i = 1, \dots, n$. The n -tuple of maps (ϕ_1, \dots, ϕ_n) is called the rigging, and is usually subject to some additional regularity conditions which vary in the conformal field theory literature. The choice of these regularity conditions relates directly to the analytic structure of this moduli space. The regularity also relates directly to the regularity of certain elliptic operators, which are necessary for the rigorous definition of conformal field theory in the sense of Segal [26]. In this paper we show that the rigged moduli space has a Hilbert manifold structure, and that this Hilbert manifold structure arises naturally from a refined Teichmüller space of bordered surfaces, which we also show is a Hilbert manifold. These results are further motivated by the fact that the aforementioned elliptic operators will have convergent determinants on precisely this refined moduli space. We hope to return to this question in a future publication. Moreover, these results will have applications to the construction of higher genus conformal field theory, following a program of Yi-Zhi Huang and others [15, 16]. Also, it is natural to ask whether there is a convergent natural generalization of the Weil-Petersson metric on the refined Teichmüller space, as in [27]. We intend to demonstrate this in a future publication.

These results are made possible by previous work of two of the authors [24], in which it was shown that if one chooses the riggings to be extendible to quasiconformal maps of a neighborhood of the closure of \mathbb{D} , then the rigged moduli space is the same as the Teichmüller space of a bordered Riemann surface (up to a properly discontinuous group action). Thus the rigged moduli space inherits a complex Banach manifold structure from Teichmüller space.

This solved certain analytic problems in the definition of conformal field theory, including holomorphicity of the sewing operation.

On the other hand this also provided an alternate description of the Teichmüller space of a bordered surface Σ as a fibre space that is locally modeled on the following rigged Teichmüller space. In [25] (following the first author's thesis [22]), two of the authors introduced the *rigged Teichmüller space* based on quasiconformally extendible riggings, which is the analogue of the above rigged moduli space. It was proved that this rigged Teichmüller space is a fibre space: the fibres consist of non-overlapping maps into a compact Riemann surface with punctures obtained by sewing copies of the punctured disc onto the boundaries of Σ . The base space is the finite-dimensional Teichmüller space of the compact surface with punctures so obtained.

Thus the Teichmüller space of bordered surfaces has two independent complex Banach manifolds structures: the standard one, obtained from the Bers embedding of spaces of equivalent Beltrami differentials, and one obtained from the fibre model. It was shown that the two are equivalent [24, 25]. Up to normalizations, the fibres look locally like an n -fold product of the universal Teichmüller space. We now define a *refined* rigged Teichmüller space and prove that it is a Hilbert manifold by using the results of Guo Hui [14] and Takhtajan and Teo [27] to define a refined set of fibres that are modeled on Hilbert spaces. Finally, we define a refined Teichmüller space of bordered surfaces and, via the fibre model, show that it is a Hilbert manifold using the refined rigged Teichmüller space. Charts for the refined Teichmüller space will be defined completely explicitly, using Gardiner-Schiffer variation and natural function spaces of non-overlapping maps.

The proof that these charts define a Hilbert manifold structure is somewhat complicated. We proceed in the following way. In Section 2, we define the refined quasiconformal mappings and function spaces which will appear in the paper. This section mostly establishes notation and outlines some previous results, and proves some elementary facts about the refined mappings. The difficult work is done in Sections 3 and 4. In Section 3, we define the set of non-overlapping mappings which serves as a model of the fibres, and show that it is a complex Hilbert manifold. In Section 4, we show that the refined rigged Teichmüller space is a Hilbert manifold. We do this using the results of the previous section, and Gardiner-Schiffer variation. A key part of the argument relies on the universality properties of the universal Teichmüller curve and the theory of marked holomorphic families of Riemann surfaces. Finally, in Section 5 we show that the refined Teichmüller space of a bordered Riemann surface is a Hilbert manifold, by showing that it covers the refined rigged Teichmüller space and passing the structure upwards. Furthermore, we show that the Hilbert manifold structure passes downwards to the two versions of the rigged moduli space of conformal field theory defined by Segal [26] and Vafa [29].

2. REFINED QUASICONFORMAL MAPS AND QUASISYMMETRIES

In Section 2.1 we collect some known results on the refinement of the set of quasimaps and quasiconformal maps, from the work of Takhtajan and Teo [27], Teo [28] and Guo Hui [14]. We also derive two technical lemmas which follow almost directly from previous work of two of the authors [24]. In Section 2.2 we define a refined set of quasimaps between borders of Riemann surfaces in an obvious way and some elementary results are derived. This is then used to define a refined set of quasiconformal maps between Riemann surfaces in Section 2.3.

2.1. Refined maps on the disc and circle. In this section we collect some necessary results on the refined universal Teichmüller space of Takhtajan and Teo [27] and Guo Hui [14]. We need a refined class of quasiconformal and quasisymmetric mappings of the disc and S^1 .

In [24] we defined the set \mathcal{O}^{qc} of quasiconformally extendible maps in the following way.

Definition 2.1. Let \mathcal{O}^{qc} be the set of maps $f : \mathbb{D} \rightarrow \mathbb{C}$ such that f is one-to-one, holomorphic, has quasiconformal extension to \mathbb{C} , and $f(0) = 0$.

A Banach space structure can be introduced on \mathcal{O}^{qc} as follows. Let

$$(2.1) \quad A_1^\infty(\mathbb{D}) = \left\{ \phi \in \mathcal{H}(\mathbb{D}) : \|\phi\|_1^\infty = \sup_{z \in \mathbb{D}} (1 - |z|^2) |\phi(z)| < \infty \right\}.$$

This is a Banach space. It follows directly from results of Teo [28] that for

$$\mathcal{A}(f) = \frac{f''}{f'}$$

the map

$$(2.2) \quad \begin{aligned} \chi : \mathcal{O}^{\text{qc}} &\longrightarrow A_1^\infty(\mathbb{D}) \oplus \mathbb{C} \\ f &\longmapsto (\mathcal{A}(f), f'(0)) \end{aligned}$$

takes \mathcal{O}^{qc} onto an open subset of the Banach space $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$ (see [24]). Thus \mathcal{O}^{qc} inherits a complex structure from $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$.

The space \mathcal{O}^{qc} can be thought of as a two complex dimensional extension of the universal Teichmüller space. We will construct a Hilbert structure on a subset of \mathcal{O}^{qc} . To do this, in place of $A_1^\infty(\mathbb{D})$ we use the Bergman space

$$A_1^2(\mathbb{D}) = \left\{ \phi \in \mathcal{H}(\mathbb{D}) : \|\phi\|_2^2 = \iint_{\mathbb{D}} |\phi|^2 dA < \infty \right\}$$

which is a Hilbert space and a vector subspace of the Banach space $A_1^\infty(\mathbb{D})$. Furthermore, the inclusion map from $A_1^2(\mathbb{D})$ to $A_1^\infty(\mathbb{D})$ is bounded [27, Chapter II Lemma 1.3]. Here and in the rest of the paper we shall denote the Bergman space norm $\|\cdot\|_{A_1^2}$ by $\|\cdot\|$.

We define the class of refined quasiconformally extendible maps as follows.

Definition 2.2. Let

$$\mathcal{O}_0^{\text{qc}} = \{f \in \mathcal{O}^{\text{qc}} : \mathcal{A}(f) \in A_1^2(\mathbb{D})\}.$$

We will embed $\mathcal{O}_0^{\text{qc}}$ in the Hilbert space direct sum $\mathcal{W} = A_1^2(\mathbb{D}) \oplus \mathbb{C}$. Since $\chi(\mathcal{O}^{\text{qc}})$ is open, $\chi(\mathcal{O}_0^{\text{qc}}) = \chi(\mathcal{O}^{\text{qc}}) \cap A_1^2(\mathbb{D})$ is also open, and thus $\mathcal{O}_0^{\text{qc}}$ trivially inherits a Hilbert manifold structure from \mathcal{W} . We summarize this with the following theorem.

Theorem 2.3. *The inclusion map from $A_1^2(\mathbb{D}) \rightarrow A_1^\infty(\mathbb{D})$ is continuous. Furthermore $\chi(\mathcal{O}_0^{\text{qc}})$ is an open subset of the vector subspace $\mathcal{W} = A_1^2(\mathbb{D}) \oplus \mathbb{C}$ of $A_1^\infty(\mathbb{D}) \oplus \mathbb{C}$, and the inclusion map from $\chi(\mathcal{O}_0^{\text{qc}})$ to $\chi(\mathcal{O}^{\text{qc}})$ is holomorphic. Thus the inclusion map $\iota : \mathcal{O}_0^{\text{qc}} \rightarrow \mathcal{O}^{\text{qc}}$ is holomorphic.*

Remark 2.4. Although the inclusion map is continuous, the topology of $\mathcal{O}_0^{\text{qc}}$ is not the relative topology inherited from \mathcal{O}^{qc} . It's enough to show that $A_1^2(\mathbb{D})$ does not have the relative topology from $A_1^\infty(\mathbb{D})$. To see this observe that if

$$f_t = \frac{1}{\sqrt{|\log(1-t)|}(1-t^2 z^2)}$$

for $t < 1$, then as $t \rightarrow 1$ $\|f_t\| \rightarrow 0$ in $A_1^2(\mathbb{D})$ whereas $\|f_t\|_{A_1^\infty(\mathbb{D})} \rightarrow \pi/2$.

Lemma 2.5. *Let $f \in \mathcal{O}_0^{\text{qc}}$. Let h be a one-to-one holomorphic map defined on an open set W containing $\overline{f(\mathbb{D})}$. Then $h \circ f \in \mathcal{O}_0^{\text{qc}}$. Furthermore, there is an open neighborhood U of f in $\mathcal{O}_0^{\text{qc}}$ and a constant C such that $\|\mathcal{A}(h \circ g)\| \leq C$ for all $g \in U$.*

Proof. The map $h \circ f$ has a quasiconformal extension to \mathbb{C} if and only if it has a quasiconformal extension to an open neighborhood of $\overline{\mathbb{D}}$ (although not necessarily with the same dilatation constant). Clearly $h \circ f$ has a quasiconformal extension to W , namely h composed with the extension of f . Thus $h \circ f$ has an extension to the plane, and so $h \circ f \in \mathcal{O}^{\text{qc}}$.

We need only show that $\mathcal{A}(h \circ f) \in A_1^2(\mathbb{D})$. This follows from Minkowski's inequality:

$$(2.3) \quad \left(\iint_{\mathbb{D}} |\mathcal{A}(h \circ f)|^2 dA \right)^{1/2} \leq \left(\iint_{\mathbb{D}} |\mathcal{A}(h) \circ f \cdot f'|^2 dA \right)^{1/2} + \left(\iint_{\mathbb{D}} |\mathcal{A}(f)|^2 dA \right)^{1/2} \\ = \left(\iint_{f(\mathbb{D})} |\mathcal{A}(h)|^2 dA \right)^{1/2} + \left(\iint_{\mathbb{D}} |\mathcal{A}(f)|^2 dA \right)^{1/2}$$

The first term on the right hand side is finite because h is holomorphic and $h' \neq 0$ on an open set containing $\overline{f(\mathbb{D})}$ so $\mathcal{A}(h)$ is bounded on $f(\mathbb{D})$. The second term is bounded because $f \in \mathcal{O}_0^{\text{qc}}$. This proves the first claim.

To prove the second claim, observe that there is a compact set K contained in W which contains $\overline{f(\mathbb{D})}$ in its interior. By [24, Corollary 3.5] there is an open set \hat{U} in \mathcal{O}^{qc} such that $\overline{g(\mathbb{D})}$ is contained in the interior of K for all $g \in \hat{U}$. Since the inclusion $\iota : \mathcal{O}_0^{\text{qc}} \rightarrow \mathcal{O}^{\text{qc}}$ is continuous, we obtain an open set $\iota^{-1}(\hat{U}) \subset \mathcal{O}_0^{\text{qc}}$ with the same property. Let U be an open ball in $\iota^{-1}(\hat{U})$ containing f . There is a constant C_1 such that for any $g \in U$

$$\iint_{\mathbb{D}} |\mathcal{A}(g)|^2 dA \leq C_1$$

and a constant C_2 such that

$$\iint_{g(\mathbb{D})} |\mathcal{A}(h)|^2 dA \leq \iint_K |\mathcal{A}(h)|^2 dA \leq C_2.$$

Applying (2.3) completes the proof. \square

We will also need a technical lemma on a certain kind of holomorphicity of left composition in $\mathcal{O}_0^{\text{qc}}$.

Lemma 2.6. *Let E be an open subset of \mathbb{C} containing 0 and Δ an open subset of \mathbb{C} . Let $\mathcal{H} : \Delta \times E \rightarrow \mathbb{C}$ be a map which is holomorphic in both variables and let $h_\epsilon(z) = \mathcal{H}(\epsilon, z)$. Let $\psi \in \mathcal{O}_0^{\text{qc}}$ satisfy $\overline{\psi(\mathbb{D})} \subseteq E$. Then the map $Q : \Delta \rightarrow \mathcal{O}_0^{\text{qc}}$ defined by $Q(\epsilon) = h_\epsilon \circ \psi$ is holomorphic in ϵ .*

Proof. We need to show that for fixed ψ , $\mathcal{A}(h_\epsilon \circ \psi)$ and $(h_\epsilon \circ \psi)'(0)$ are holomorphic in ϵ . First observe that all the z -derivatives of h_ϵ are holomorphic in ϵ for fixed z . Thus the second claim is immediate.

To prove holomorphicity of $\epsilon \mapsto \mathcal{A}(h_\epsilon \circ \psi)$, it is enough to show weak holomorphicity and local boundedness [13]; that is, to show local boundedness and that for some set of separating continuous functionals $\{\alpha\}$ in the dual of the Bergman space, $\alpha \circ \mathcal{A}(h_\epsilon \circ \psi)$ is holomorphic

for all α . Let E_z be the point evaluation function $E_z\psi = \psi(z)$. These are continuous on the Bergman space and obviously separating on any open set. Since

$$\mathcal{A}(h_\epsilon \circ \psi) = \mathcal{A}(h_\epsilon) \circ \psi \cdot \psi' + \mathcal{A}(\psi)$$

clearly $E_z(\mathcal{A}(h_\epsilon \circ \psi))$ is holomorphic in ϵ .

So we only need to prove that $\mathcal{A}(h_\epsilon \circ \psi)$ and $(h_\epsilon \circ \psi)'(0)$ are locally bounded. The second claim is obvious. As above, by Minkowski's inequality (2.3) and a change of variables

$$\left(\iint_{\mathbb{D}} |\mathcal{A}(h_\epsilon \circ \psi)|^2 dA \right)^{1/2} \leq \left(\iint_{\psi(\mathbb{D})} |\mathcal{A}(h_\epsilon)|^2 dA \right)^{1/2} + \left(\iint_{\mathbb{D}} |\mathcal{A}(\psi)|^2 dA \right)^{1/2}.$$

Since $\mathcal{A}(h_\epsilon)$ is jointly holomorphic in ϵ and z and $\overline{\psi(\mathbb{D})} \subseteq E$ for any fixed ϵ_0 , there is a compact set D containing ϵ_0 such that $|\mathcal{A}(h_\epsilon)|$ is bounded on $\psi(\mathbb{D})$ by a constant independent of $\epsilon \in D$. Since $\mathcal{A}(\psi)$ is in the Bergman space this proves the claim. \square

Next, we define a subset $\text{QS}_0(S^1)$ of the quasimorphisms in the following way. Briefly, a map $h : S^1 \rightarrow S^1$ is in $\text{QS}_0(S^1)$ if the corresponding welding maps are in $\mathcal{O}_0^{\text{qc}}$. Let $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$. For $h \in \text{QS}(S^1)$ let $w_\mu(h) : \mathbb{D}^* \rightarrow \mathbb{D}^*$ be a quasiconformal extension of h with dilatation μ (such an extension exists by the Ahlfors-Beurling extension theorem). Furthermore, let $w^\mu : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ be the quasiconformal map with dilatation μ on \mathbb{D}^* and 0 on \mathbb{D} , with normalization $w^\mu(0) = 0$, $w^{\mu'}(0) = 1$ and $w^\mu(\infty) = \infty$ and set

$$F(h) = w^\mu|_{\mathbb{D}}.$$

It is a standard fact that $F(h)$ is independent of the choice of extension w_μ .

Definition 2.7. We define a subset of $\text{QS}(S^1)$ by

$$\text{QS}_0(S^1) = \{h \in \text{QS}(S^1) : F(h) \in \mathcal{O}_0^{\text{qc}}\}.$$

Remark 2.8. A change in the normalization of $w^{\mu'}(0)$ results in exactly the same set.

An alternate characterization of $\mathcal{O}_0^{\text{qc}}$ follows from a theorem proved by Guo Hui [14]. Let

$$L_{\text{hyp}}^2(\mathbb{D}^*) = \left\{ \mu : \iint_{\mathbb{D}^*} (|z|^2 - 1)^{-2} |\mu(z)|^2 dA < \infty \right\},$$

and let

$$L^\infty(\mathbb{D}^*)_1 = \{\mu : \mathbb{D}^* \rightarrow \mathbb{C} : \|\mu\|_\infty \leq k \text{ for some } k < 1\}$$

(that is, the unit ball in $L^\infty(\mathbb{D}^*)$). Note that the line element of the hyperbolic metric on \mathbb{D} is $|dz|(1 - |z|^2)^{-1}$ and the line element of the hyperbolic metric on \mathbb{D}^* is $|dz|(|z|^2 - 1)^{-1}$. Thus the above condition says that μ is L^2 with respect to hyperbolic area. The following two theorems follow from Theorems 1 and 2 of [14].

Theorem 2.9 (Guo Hui). *Let f be a one-to-one holomorphic function on \mathbb{D} such that $f(0) = 0$. Then $f \in \mathcal{O}_0^{\text{qc}}$ if and only if there exists a quasiconformal extension \tilde{f} of f to \mathbb{C} whose dilatation μ is in $L_{\text{hyp}}^2(\mathbb{D}^*) \cap L^\infty(\mathbb{D}^*)_1$.*

Theorem 2.10 (Guo Hui). *Let $\phi : S^1 \rightarrow S^1$ be a quasimorphism. Then $\phi \in \text{QS}_0(S^1)$ if and only if there is a quasiconformal extension $h : \mathbb{D}^* \rightarrow \mathbb{D}^*$ of ϕ such that the Beltrami differential $\mu(h)$ of h is in $L_{\text{hyp}}^2(\mathbb{D}^*)$.*

It follows from Theorem 1.12 of Part II and Lemma 3.4 of Part I of [27] that $\text{QS}_0(S^1)$ is a group.

Theorem 2.11 (Takhtajan-Teo). *The set $QS_0(S^1)$ is closed under composition and inversion.*

By an analytic map $h : S^1 \rightarrow S^1$ we mean that h is the restriction of an analytic map of a neighborhood of S^1 . Let $\mathbb{A}(r, s)$ denote the annulus $\{z : r < |z| < s\}$ and $D(z_0, r)$ denote the disc $\{z : |z - z_0| < r\}$.

Proposition 2.12. *If $h : S^1 \rightarrow S^1$ is one-to-one and analytic, then h has a quasiconformal extension to \mathbb{D}^* which is holomorphic in an annulus $\mathbb{A}(1, R)$ for some $R > 1$. Furthermore $h \in QS_0(S^1)$.*

Proof. To prove the first claim, observe that h has an analytic extension \tilde{h} to some annulus $\mathbb{A}(r, s)$ for $r < 1 < s$. Let R be such that $1 < R < s$. Applying the Ahlfors-Beurling extension theorem to the circle $|z| = R$, there exists a quasiconformal map $g : \mathbb{A}(R, \infty) \rightarrow \mathbb{A}(R, \infty)$ whose boundary values agree with \tilde{h} restricted to $|z| = R$. Let H be the map which is equal to \tilde{h} on $\mathbb{A}(1, R)$ and g on $\mathbb{A}(R, \infty)$. Then H is quasiconformal on \mathbb{D}^* since it is quasiconformal on the two pieces and continuous on \mathbb{D} (see [18, V.3]). Thus, H has the desired properties.

The second claim follows from Theorem 2.9 since the dilatation of H is zero in $\mathbb{A}(1, R)$. \square

2.2. Refined quasisymmetric mappings between boundaries of Riemann surfaces.

We first clarify the meaning of “bordered Riemann surface”. By a half-disc, we mean a set of the form $\{z : |z - z_0| < r \text{ and } \text{Im}(z) \geq 0\}$ for some z_0 on the real axis. By a bordered Riemann surface, we mean a Riemann surface with boundary, such that for every point on the boundary there is a homeomorphism of a neighborhood of that point onto a half-disc. It is further assumed that for any pair of charts ρ_1, ρ_2 whose domains overlap, the map $\rho_2 \circ \rho_1^{-1}$ and its inverse is a one-to-one holomorphic map on its domain. Note that this implies, by the Schwarz reflection principle, that $\rho_2 \circ \rho_1^{-1}$ extends to a one-to-one holomorphic map of an open set containing the portion of the real axis in the domain of the original map. Every bordered Riemann surface has a double which is defined in the standard way. See for example [1].

Following standard terminology (see for example [21]) we say that a Riemann surface is of *finite topological type* if its fundamental group is finitely generated. A Riemann surface is said to be of finite topological type (g, n, m) if it is biholomorphic to a compact genus g Riemann surface with n points and m parametric disks removed. By a parametric disk we mean a region biholomorphic to the unit disk such that, after its removal, the resultant surface is homeomorphic to a compact surface with a point removed.

In this paper we will be entirely concerned with Riemann surfaces of type $(g, 0, n)$ and $(g, n, 0)$ and we will use the following terminology. A *bordered Riemann surface* of type (g, n) will refer to a bordered Riemann surface of type $(g, 0, n)$ and a *punctured Riemann surface* of type (g, n) will refer to a Riemann surface of type $(g, n, 0)$. It is furthermore assumed that the boundary curves and punctures are given a numerical ordering. Finally, a *boundary curve* will be understood to mean a connected component of the boundary of a bordered Riemann surface. Note that each boundary curve is homeomorphic to S^1 .

Remark 2.13. Any quasiconformal map between bordered Riemann surfaces has a unique continuous extension taking the boundary curves to the boundary curves. To see this let Σ_1^B and Σ_2^B be bordered Riemann surfaces, and let Σ_1^d and Σ_2^d denote their doubles. By reflecting, the quasiconformal map extends to the double: the reflected map is continuous on Σ_1^d , takes Σ_1^d onto Σ_2^d , and is quasiconformal on the double minus the boundary curves. Since each

boundary curve of Σ_i^B is an analytic curve in the double, the map is quasiconformal on Σ_1^d [18, V.3] and in particular continuous on each analytic curve.

Throughout the paper, we will label the original map and its continuous extension with the same letter to avoid complicating the notation. When referring to a “bordered Riemann surface”, we will be referring to the interior. However, in the following all maps between bordered Riemann surfaces will be at worst quasiconformal and thus by Remark 2.13 have unique continuous extensions to the boundary. Thus the reader could treat the border as included in the Riemann surface with only trivial changes to the statements in the rest of the paper.

Definition 2.14. Let Σ^B be a bordered Riemann surface and C be one of its boundary components. A *collared neighborhood* of C is an open set U which is biholomorphic to an annulus, and one of whose boundary curves is C . A *collared chart* of C is a biholomorphism $H : U \rightarrow \mathbb{A}(1, r)$ where U is a collared neighborhood of C , whose continuous extension to C maps C to S^1 .

Note that any collared chart must have a continuous one-to-one extension to C , which maps C to S^1 . (In fact application of the Schwarz reflection principle shows that H must have a one-to-one holomorphic extension to an open tubular neighborhood of C in the double of Σ .) We may now define the class of refined quasisymmetries between boundary curves of bordered Riemann surfaces.

Definition 2.15. Let Σ_1^B and Σ_2^B be bordered Riemann surfaces, and let C_1 and C_2 be boundary curves of Σ_1^B and Σ_2^B respectively. Let $\text{QS}_0(C_1, C_2)$ denote the set of orientation-preserving homeomorphisms $\phi : C_1 \rightarrow C_2$ such that there are collared charts H_i of C_i , $i = 1, 2$ respectively, such that $H_2 \circ \phi \circ H_1^{-1}|_{S^1} \in \text{QS}_0(S^1)$.

Remark 2.16. The notation $\text{QS}_0(S^1, C_1)$ will always be understood to refer to S^1 as the boundary of an annulus $\mathbb{A}(1, r)$ for $r > 1$. We will also write $\text{QS}_0(S^1) = \text{QS}_0(S^1, S^1)$.

Proposition 2.17. *If $\phi \in \text{QS}_0(C_1, C_2)$ then for any pair of collared charts H_i of C_i , $i = 1, 2$ respectively, $H_2 \circ \phi \circ H_1^{-1}|_{S^1} \in \text{QS}_0(S^1)$.*

Proof. Assume that there are collared charts H'_i of C_i such that $H'_2 \circ \phi \circ H_1'^{-1} \in \text{QS}_0(S^1)$. Let H_i be any other pair of collared charts. The composition

$$H_2 \circ H_2'^{-1} \circ H_2' \circ \phi \circ H_1'^{-1} \circ H_1' \circ H_1^{-1} = H_2 \circ \phi \circ H_1^{-1}$$

is defined on some collared neighborhood of C_1 . Since $H_2 \circ H_2'^{-1}$ and $H_1' \circ H_1^{-1}$ have analytic extensions to S^1 , the result follows from Proposition 2.12 and Theorem 2.11. \square

Proposition 2.18. *Let Σ_i^B be bordered Riemann surfaces and C_i a boundary curve on each surface for $i = 1, 2, 3$. If $\phi \in \text{QS}_0(C_1, C_2)$ and $\psi \in \text{QS}_0(C_2, C_3)$ then $\psi \circ \phi \in \text{QS}_0(C_1, C_3)$.*

Proof. Let H_i be collared charts of C_i for $i = 1, 2, 3$. In that case

$$H_3 \circ \psi \circ \phi \circ H_1^{-1} = H_3 \circ \psi \circ H_2^{-1} \circ H_2 \circ \phi \circ H_1^{-1}$$

when restricted to C_1 . By Proposition 2.17 both $H_3 \circ \psi \circ H_2^{-1}$ and $H_2 \circ \phi \circ H_1^{-1}$ are in $\text{QS}_0(S^1)$, so the composition is in $\text{QS}_0(S^1)$ by Theorem 2.11. Thus $\psi \circ \phi \in \text{QS}_0(C_1, C_3)$ by definition. \square

2.3. A refined class of quasiconformal mappings between bordered surfaces. We can now define a refined class of quasiconformal mappings.

Definition 2.19. Let Σ_1^B and Σ_2^B be bordered Riemann surfaces of type (g, n) , with boundary curves C_1^i and C_2^j $i = 1, \dots, n$ and $j = 1, \dots, n$ respectively. The class of maps $\text{QC}_0(\Sigma_1^B, \Sigma_2^B)$ consists of those quasiconformal maps from Σ_1^B onto Σ_2^B such that the continuous extension to each boundary curve C_1^i , $i = 1, \dots, n$ is in $\text{QS}_0(C_1^i, C_2^j)$ for some $j \in \{1, \dots, n\}$.

Note that the continuous extension to a boundary curve C_1^i *must* map onto a boundary curve C_2^j .

The following two Propositions follow immediately from Definition 2.19 and Proposition 2.18.

Proposition 2.20. Let Σ_i^B $i = 1, 2, 3$ be bordered Riemann surfaces of type (g, n) . If $f \in \text{QC}_0(\Sigma_1^B, \Sigma_2^B)$ and $g \in \text{QC}_0(\Sigma_2^B, \Sigma_3^B)$ then $g \circ f \in \text{QC}_0(\Sigma_1^B, \Sigma_3^B)$.

Proposition 2.21. Let Σ_1^B and Σ_2^B be bordered Riemann surfaces. Let C_1 be a boundary curve of Σ_1^B , $\phi \in \text{QS}_0(S^1, C_1)$, $f \in \text{QC}_0(\Sigma_1^B, \Sigma_2^B)$ and $C_2 = f(C_1)$ be the boundary curve of Σ_2^B onto which f maps C_1 . Then $f \circ \phi \in \text{QS}_0(S^1, C_2)$.

3. NON-OVERLAPPING MAPPINGS

In this section we show that the class of non-overlapping holomorphic maps into a Riemann surface, with refined quasiconformal extensions, is a Hilbert manifold. The class of non-overlapping mappings is the infinite-dimensional part of both the moduli space of Friedan and Shenker and the refined Teichmüller space.

Let Σ be a punctured Riemann surface of type (g, n) . In Section 3.1, we define the class of non-overlapping mappings $\mathcal{O}_0^{\text{qc}}(\Sigma)$ and establish a technical theorem which is central to the proof that it is a Hilbert manifold. Section 3.2 is devoted to defining a topology and atlas on $\mathcal{O}_0^{\text{qc}}(\Sigma)$, and the proof that this topology is Hausdorff, second countable, and the overlap maps of the atlas are biholomorphisms.

3.1. Definitions and technical results. We define a class of non-overlapping mappings into a punctured Riemann surface. Let \mathbb{D}_0 denote the punctured disc $\mathbb{D} \setminus \{0\}$. Let Σ be a compact Riemann surface with punctures p_1, \dots, p_n .

Definition 3.1. The class of non-overlapping quasiconformally extendible maps $\mathcal{O}^{\text{qc}}(\Sigma)$ into Σ is the set of n -tuples (ϕ_1, \dots, ϕ_n) where

- (1) For all $i \in \{1, \dots, n\}$, $\phi_i : \mathbb{D}_0 \rightarrow \Sigma$ is holomorphic, and has a quasiconformal extension to a neighborhood of $\overline{\mathbb{D}}$.
- (2) The continuous extension of ϕ_i takes 0 to p_i
- (3) For any $i \neq j$, $\overline{\phi_i(\mathbb{D})} \cap \overline{\phi_j(\mathbb{D})}$ is empty.

It was shown in [24] that $\mathcal{O}^{\text{qc}}(\Sigma)$ is a complex Banach manifold.

As in the previous section, we need to refine the class of non-overlapping mappings. We first introduce some terminology. Denote the compactification of a punctured surface Σ by $\overline{\Sigma}$.

Definition 3.2. An n -chart on Σ is a collection of open sets E_1, \dots, E_n contained in the compactification of Σ such that $E_i \cap E_j$ is empty whenever $i \neq j$, together with local parameters $\zeta_i : E_i \rightarrow \mathbb{C}$ such that $\zeta_i(p_i) = 0$.

In the following, we will refer to the charts (ζ_i, E_i) as being on Σ , with the understanding that they are in fact defined on the compactification. Similarly, non-overlapping maps (f_1, \dots, f_n) will be extended by the removable singularities theorem to the compactification, without further comment.

Definition 3.3. Let $\mathcal{O}_0^{\text{qc}}(\Sigma)$ be the set of n -tuples of maps $(f_1, \dots, f_n) \in \mathcal{O}^{\text{qc}}(\Sigma)$ such that for any choice of n -chart $\zeta_i : E_i \rightarrow \mathbb{C}$, $i = 1, \dots, n$ satisfying $\overline{f_i(\mathbb{D})} \subset E_i$ for all $i = 1, \dots, n$, it holds that $\zeta_i \circ f_i \in \mathcal{O}_0^{\text{qc}}$.

The space $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is well-defined. To see this let (ζ_i, E_i) and (η_i, F_i) , $i = 1, \dots, n$, be n -charts satisfying $\overline{f_i(\mathbb{D})} \subset E_i \cap F_i$ and assume that $\zeta_i \circ f_i \in \mathcal{O}_0^{\text{qc}}$. Since $\eta_i \circ \zeta_i^{-1}$ is holomorphic on an open set containing $\zeta_i \circ f_i(\mathbb{D})$, it follows from Lemma 2.5 that $\eta_i \circ f_i = \eta_i \circ \zeta_i^{-1} \circ \zeta_i \circ f_i \in \mathcal{O}_0^{\text{qc}}$.

In order to construct a Hilbert manifold structure on $\mathcal{O}_0^{\text{qc}}(\Sigma)$ we will need some technical theorems.

Theorem 3.4. *Let E be an open neighborhood of 0 in \mathbb{C} . Then the set*

$$\left\{ f \in \mathcal{O}^{\text{qc}} : \overline{f(\mathbb{D})} \subset E \right\}$$

is open in \mathcal{O}^{qc} and the set

$$\left\{ f \in \mathcal{O}_0^{\text{qc}} : \overline{f(\mathbb{D})} \subset E \right\}$$

is open in $\mathcal{O}_0^{\text{qc}}$.

Proof. Let $f_0 \in \mathcal{O}^{\text{qc}}$ satisfy $\overline{f_0(\mathbb{D})} \subset E$. By [24, Corollary 3.5], there exists an open subset W of \mathcal{O}^{qc} such that $\overline{f(\mathbb{D})} \subset E$ for all $f \in W$. Since f_0 was arbitrary, this proves the first claim.

Now let $f_0 \in \mathcal{O}_0^{\text{qc}}$ satisfy $\overline{f_0(\mathbb{D})} \subset E$. As above, there exists an open subset W of \mathcal{O}^{qc} such that $\overline{f(\mathbb{D})} \subset E$ for all $f \in W$. But by Theorem 2.3 $W \cap \mathcal{O}_0^{\text{qc}} = \iota^{-1}(W)$ is open in $\mathcal{O}_0^{\text{qc}}$. Thus $\overline{f(\mathbb{D})} \subset E$ for all f in the open set $W \cap \mathcal{O}_0^{\text{qc}}$ containing f_0 . This proves the second claim. \square

Composition on the left by h is holomorphic operation in both \mathcal{O}^{qc} and $\mathcal{O}_0^{\text{qc}}$. This was proven in [24] in the case of \mathcal{O}^{qc} . The corresponding theorem in the refined case is considerably more delicate, and is one of the key theorems necessary to demonstrate the existence of a Hilbert manifold structure on $\mathcal{O}_0^{\text{qc}}(\Sigma^P)$. Before we state and prove it we need to investigate some purely analytic issues in the underlying function theory, which will be utilized later.

We start first with the following lemma.

Lemma 3.5. *Let $f_t(z)$ be a holomorphic curve in $\mathcal{O}_0^{\text{qc}}$ for $t \in N$ where $N \subset \mathbb{C}$ is an open set containing 0. Then there is a domain $N' \subseteq N$ containing 0 and a K which is independent of $t \in N'$ such that*

$$(3.1) \quad \iint_{\mathbb{D}} |f'_t(z)|^p (1 - |z|^2)^\alpha dA \leq K,$$

for all $p > 0$ and $\alpha > -1$. The constant K will depend on p and α .

Proof. To establish the estimate (3.1) we observe that since $\mathcal{A}(f_t) \in A_1^2(\mathbb{D})$, $\log f'_t$ is in the little Bloch space; that is

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) |g'_t(z)| = 0,$$

see [27, Corollary 1.4, Chapter 2]. By [12, Theorem 1 (1)], the integral in (3.1) is finite for each t . However, we need a uniform estimate in t . Although this does not follow from the

theorem as stated in [12, Theorem 1 (1)], the proof of that theorem can be modified to get the uniform estimate. We proceed by providing the details of this argument. The claim of [12, Theorem 1 (1)] is that

$$(3.2) \quad g = \log f' \in \mathcal{B}_0 \implies \iint_{\mathbb{D}} |f'|^p (1 - |z|^2)^\alpha dA < \infty$$

for all $p > 0$ and $\alpha > -1$ where \mathcal{B}_0 is the little Bloch space.

Let $h_s(z) = g(sz)$. This function is continuous on $\overline{\mathbb{D}}$ for $0 < s < 1$. Hence for each fixed s the integral in question converges by an elementary estimate. Therefore (3.2) will follow if we can show that the integral is uniformly bounded for s in some interval $[s_0, 1)$.

We have that $h_s \in \mathcal{B}_0$, that is,

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) |h'_s(z)| = 0$$

for all $0 < s \leq 1$. Since $h_1(z) = g(z)$ is in the little Bloch space, and S^1 is compact, given any $\epsilon > 0$ there is an $R > 0$ such that $(1 - |z|^2) |h'_1(z)| < \epsilon$ for all $|z| > R$. Fix any $0 < s_0 < 1$ and let $r = R/s_0$. Therefore, if $|z| > r$ and $s_0 < s \leq 1$ then $|sz| > s_0 r = R$ and so for all $|z| > r$ and $s_0 < s \leq 1$ we have $(1 - |z|^2) |h'_s(z)| = (1 - |z|^2) s |h'_1(sz)| < s\epsilon \leq \epsilon$.

Thus for any $\epsilon > 0$ there are fixed $0 < r < 1$ and $0 < s_0 < 1$ such that

$$(3.3) \quad (1 - |z|^2) |h'_s(z)| < \epsilon$$

for all $(s, z) \in [s_0, 1] \times \overline{\mathbb{D}} \setminus D_r$ where $D_r = \{z : |z| < r\}$. Now set

$$\begin{aligned} I &= \iint_{\mathbb{D}} |e^{h_s(z)}|^p (1 - |z|^2)^\alpha dA, \\ I_1 &= \iint_{D_r} |e^{h_s(z)}|^p (1 - |z|^2)^\alpha dA, \\ I_2 &= \iint_{\mathbb{D} \setminus D_r} |e^{h_s(z)}|^p (1 - |z|^2)^\alpha dA. \end{aligned}$$

Our goal is to show that there is a constant C which is independent of $s \in [s_0, 1)$ such that I is bounded by C . It is obvious that this will follow by establishing the aforementioned type of bounds for I_1 and I_2 . The estimate for I_1 follows from

$$\begin{aligned} (3.4) \quad \iint_{D_r} |e^{h_s(z)}|^p (1 - |z|^2)^\alpha dA &\leq \frac{(1 - r^2)^{\min(\alpha, 0)}}{s^2} \iint_{D_{rs}} |e^{h_1(z)}|^p dA \\ &\leq \frac{(1 - r^2)^{\min(\alpha, 0)}}{s_0^2} \iint_{D_r} |e^{h_1(z)}|^p dA \\ &\leq C. \end{aligned}$$

Now we turn to the estimate for I_2 . It follows from a theorem of Hardy and Littlewood (see for example [8, Theorem 6] for a proof in the most general case) that there is a C depending only on p and α , such that

$$(3.5) \quad \iint_{\mathbb{D}} |F(z)|^p (1 - |z|^2)^\alpha dA \leq C \left(\iint_{\mathbb{D}} |F'(z)|^p (1 - |z|^2)^{p+\alpha} dA + |F(0)|^p \right)$$

for $p > 0$ and $\alpha > -1$, whenever at least one of the integrals converges (in fact the two norms represented by each side are equivalent). Now for $s \in [s_0, 1)$ we may apply (3.5) and (3.3) to

$e^{h_s(z)}$ which yield

$$\begin{aligned}
I_2 &\leq \iint_{\mathbb{D}} |e^{h_s(z)}|^p (1 - |z|^2)^\alpha dA \\
&\leq C \left(\iint_{\mathbb{D}} |e^{h_s(z)}|^p |h'_s(z)|^p (1 - |z|^2)^{p+\alpha} dA + |e^{h_s(0)}|^p \right) \\
&\leq C \iint_{\mathbb{D} \setminus D_r} |e^{h_s(z)}|^p |h'_s(z)|^p (1 - |z|^2)^{p+\alpha} dA + C \iint_{D_r} |e^{h_s(z)}|^p |h'_s(z)|^p (1 - |z|^2)^{p+\alpha} dA \\
&\quad + C |e^{h_s(0)}|^p \\
&\leq C\epsilon I_2 + C \iint_{D_r} |e^{h_s(z)}|^p |h'_s(z)|^p (1 - |z|^2)^{p+\alpha} dA + C |e^{h_s(0)}|^p \\
&\leq \frac{1}{2} I_2 + C \iint_{D_r} |e^{h_s(z)}|^p |h'_s(z)|^p (1 - |z|^2)^{p+\alpha} dA + C |e^{h_s(0)}|^p,
\end{aligned}$$

by choosing $\epsilon \leq \frac{1}{2C}$. Summarizing, we have

$$(3.6) \quad I_2 \leq 2C \left(\iint_{D_r} |e^{h_s(z)}|^p |h'_s(z)|^p (1 - |z|^2)^{p+\alpha} dA + |e^{h_s(0)}|^p \right),$$

where r and C are independent of s . Since h_s and h'_s are continuous on $\overline{D_r}$ for $s \in [s_0, 1)$ the integral on the right hand side is bounded by a constant which is independent of $s \in [s_0, 1)$. Therefore the estimates for I_1 and I_2 yield the desired uniform estimate for I . Since the estimate on I is uniform it extends to $s = 1$.

Setting $g_t = \log f'_t$, an argument identical to the above (substituting h_s with g_t) gives the desired uniform bound (3.1) in t , provided that the function $(1 - |z|^2)|g'_t(z)|$ is jointly continuous in (t, z) . Thus it remains to demonstrate the joint continuity. To this end fix $z_0 \in \overline{\mathbb{D}}$, $t_0 \in N$ and $\epsilon > 0$. There is a δ such that for any $z \in B(z_0, \delta) \cap \overline{\mathbb{D}}$ where $B(z_0, r)$ is the ball of radius δ centered on z_0 ,

$$\|(1 - |z|^2)g'_{t_0}(z) - (1 - |z|^2)g'_{t_0}(z_0)\|_\infty < \frac{\epsilon}{2}.$$

Since f_t is a holomorphic curve, there is an interval $(t_0 - \delta_1, t_0 + \delta_1)$ such that

$$\|\mathcal{A}(f_t) - \mathcal{A}(f_{t_0})\| < \epsilon/2.$$

By [27, Lemma 1.3, Chapter II] for $g = \log f'$

$$\|(1 - |z|^2)g'(z)\|_\infty \leq \frac{1}{\sqrt{\pi}} \|\mathcal{A}(f)\|$$

(note that in their notation the left hand side is $\|g'(z)\|_\infty$). So for all $z \in \mathbb{D}$ and $t \in (t_0 - \delta_1, t_0 + \delta_1)$,

$$(3.7) \quad \|(1 - |z|^2)g'_t(z) - (1 - |z|^2)g'_{t_0}(z)\|_\infty < \frac{\epsilon}{2}.$$

Combining this with the fact that $(1 - |z|^2)g'_t(z) \rightarrow 0$ as $|z| \rightarrow 1$ shows that equation (3.7) holds on $\overline{\mathbb{D}}$. Thus, by the triangle inequality

$$\|(1 - |z|^2)g'_t(z) - (1 - |z|^2)g'_{t_0}(z_0)\|_\infty < \epsilon$$

on $(t_0 - \delta_1, t_0 + \delta_1) \times (D(z_0, r) \cap \overline{\mathbb{D}})$. This proves joint continuity and thus completes the proof. \square

Before we state our next lemma we would need some tools from the theory of Besov spaces which we recall below.

Definition 3.6. For $p \in (1, \infty)$, one defines the *Besov space* B^p as the space of holomorphic functions f on \mathbb{D} for which

$$\|f\|_{B^p} = |f(0)| + \left\{ \iint_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dA \right\}^{\frac{1}{p}} < \infty.$$

From this definition it follows at once that B^2 is the usual Dirichlet space. One also defines for $z \in \mathbb{D}$, the set $S(z)$ by

$$(3.8) \quad S(z) = \left\{ \zeta \in \mathbb{D} : 1 - |\zeta| \leq 1 - |z|, \left| \frac{\arg(z \bar{\zeta})}{2\pi} \right| \leq \frac{1 - |z|}{2} \right\},$$

which is obviously a subset of the annulus $|z| \leq |\zeta| < 1$.

In our study we shall use the following result, concerning Carleson measures for Besov spaces, due to N. Arcozzi, R. Rochberg and E. Sawyer [2].

Theorem 3.7. *Given real numbers p and q with $1 < p < q < \infty$ and a positive Borel measure μ on \mathbb{D} , the following two statements are equivalent:*

- (1) *There is a constant $C(\mu) > 0$ such that*

$$\|f\|_{L^q(\mu)} \leq C(\mu) \|f\|_{B^p}.$$

- (2) *For $S(z)$ defined above, one has*

$$\mu(S(z))^{\frac{1}{q}} \leq C \left\{ \log \frac{1 + |z|}{1 - |z|} \right\}^{-\frac{1}{p'}},$$

where p' is the Hölder dual of p .

Using Lemma 3.5 and Theorem 3.7 we can prove the following result:

Lemma 3.8. *Let $f_t(z)$ be a holomorphic curve in $\mathcal{O}_0^{\text{qc}}$ for $t \in N$ where $N \subset \mathbb{C}$ is an open set containing 0. For any holomorphic function $\psi : \mathbb{D} \rightarrow \mathbb{C}$ such that $\iint_{\mathbb{D}} |\psi'|^2 < \infty$ and $\psi(0) = 0$, and any $\beta > 1$, there is a constant C and an open set $N' \subseteq N$ containing 0 such that for all $t \in N'$*

$$\iint_{\mathbb{D}} |f'_t|^2 |\psi|^\beta dA \leq C.$$

Proof. The Cauchy-Schwarz inequality and Lemma 3.5 with $p = 4$ and $\alpha = -\frac{1}{2}$ yield

$$\begin{aligned} \iint_{\mathbb{D}} |f'_t(z)|^2 |\psi(z)|^\beta dA &\leq \left\{ \iint_{\mathbb{D}} |f'_t(z)|^4 (1 - |z|^2)^{-\frac{1}{2}} dA \right\}^{\frac{1}{2}} \times \left\{ \iint_{\mathbb{D}} |\psi(z)|^{2\beta} (1 - |z|^2)^{\frac{1}{2}} dA \right\}^{\frac{1}{2}} \\ &\leq \sqrt{K} \left\{ \iint_{\mathbb{D}} |\psi(z)|^{2\beta} (1 - |z|^2)^{\frac{1}{2}} dA \right\}^{\frac{1}{2}}. \end{aligned}$$

Therefore, since ψ is in the Dirichlet space, to prove that $\iint_{\mathbb{D}} |f'_t(z)|^2 |\psi(z)|^\beta dA \leq C$, it would be enough to show that

$$(3.9) \quad \left\{ \iint_{\mathbb{D}} |\psi(z)|^{2\beta} (1 - |z|^2)^{\frac{1}{2}} dA \right\}^{\frac{1}{2\beta}} \leq C' \left\{ \iint_{\mathbb{D}} |\psi'(z)|^2 dA \right\}^{\frac{1}{2}}.$$

Now, since $\psi(0) = 0$, Theorem 3.7 with $q = 2\beta$, $p = 2$ and $d\mu = (1 - |\zeta|^2)^{\frac{1}{2}} dA$, yields that (3.9) holds if and only if for all $z \in \mathbb{D}$

$$(3.10) \quad \left\{ \iint_{S(z)} (1 - |\zeta|^2)^{\frac{1}{2}} dA \right\}^{\frac{1}{2\beta}} \leq C' \left\{ \log \frac{1 + |z|}{1 - |z|} \right\}^{-\frac{1}{2}}.$$

Moreover

$$\iint_{S(z)} (1 - |\zeta|^2)^{\frac{1}{2}} dA \leq \iint_{|z| \leq |\zeta| < 1} (1 - |\zeta|^2)^{\frac{1}{2}} dA = 4\pi \frac{(1 - |z|^2)^{\frac{3}{2}}}{3}.$$

Therefore an elementary calculation yields that (3.10) follows from an estimate of the form

$$(3.11) \quad (1 - |z|^2)^{\frac{3}{2\beta}} \log \frac{1 + |z|}{1 - |z|} \leq C,$$

for all $|z| < 1$. Now if we set $f(r) = (1 - r^2)^{\frac{3}{2\beta}} \log \frac{1+r}{1-r}$ then for all $\varepsilon > 0$, $f(r)$ is continuous on the compact interval $[0, 1 - \varepsilon]$. Indeed the continuity of $f(r)$ is obvious on $[0, 1 - \varepsilon)$ and moreover

$$\lim_{r \rightarrow 1^-} (1 - r^2)^{\frac{3}{2\beta}} \log \frac{1+r}{1-r} = 0.$$

From this, (3.11) follows and the proof of the lemma is now complete. \square

Now we will state and prove the holomorphicity of the operation of left composition in $\mathcal{O}_0^{\text{qc}}$ which will play a crucial role in the establishment of the existence of the Hilbert manifold structure on $\mathcal{O}_0^{\text{qc}}(\Sigma^P)$.

Theorem 3.9. *Let $K \subset \mathbb{C}$ be a compact set which is the closure of an open neighborhood K_{int} of 0 and let A be an open set in \mathbb{C} containing K . If U is the open set*

$$U = \{g \in \mathcal{O}_0^{\text{qc}} : \overline{g(\mathbb{D})} \subset K_{\text{int}}\},$$

and $h : A \rightarrow \mathbb{C}$ is a one-to-one holomorphic map such that $h(0) = 0$, then the map $f \mapsto h \circ f$ from U to $\mathcal{O}_0^{\text{qc}}$ is holomorphic.

Remark 3.10. The fact that U is open follows from Theorem 3.4.

Proof. It was shown in [24, Lemma 3.10] that composition on the left is holomorphic in the above sense on \mathcal{O}^{qc} . However, this does not immediately lead to the desired result, since the norm has changed. Nevertheless some of the computations in [24, Lemma 3.10] can be used here.

As in [24, Lemma 3.10], by Hartogs' theorem [19] it suffices to show that the maps $(\mathcal{A}(f), f'(0)) \mapsto \mathcal{A}(h \circ f)$ and $f'(0) \mapsto h'(0)f'(0)$ are separately holomorphic. The second map is clearly holomorphic. By a theorem in [4, p 198], it suffices to show that $(\mathcal{A}(f), f'(0)) \mapsto \mathcal{A}(h \circ f)$ is Gâteaux holomorphic and locally bounded. It is locally bounded by Lemma 2.5.

To show that this map is Gâteaux holomorphic, consider the curve $(\mathcal{A}(f_0) + t\phi, q(t))$ where $\phi \in A_1^2(\mathbb{D})$ and q is holomorphic in t with $q(0) = f'_0(0)$. It can be easily computed that

$(\mathcal{A}(f_t), f'_t(0)) = (\mathcal{A}(f_0) + t\phi, q(t))$ if and only if f_t is the curve

$$f_t(z) = \frac{q(t)}{f'_0(0)} \int_0^z f'_0(u) \exp \left(t \int_0^u \phi(w) dw \right) du.$$

Note that $f_t(z)$ is holomorphic in t for fixed z . Since $\chi(\mathcal{O}^{\text{qc}})$ is open and $\iota : \mathcal{O}_0^{\text{qc}} \rightarrow \mathcal{O}^{\text{qc}}$ is continuous, there is an open neighborhood N of 0 in \mathbb{C} such that $f_t \in \mathcal{O}_0^{\text{qc}}$ for all $t \in N$. The neighborhood N can also be chosen small enough that $\overline{f_t(\mathbb{D})} \subset K_{\text{int}}$ for all $t \in N$, since we assumed that $t \mapsto f_t$ is a holomorphic curve and the set of $f \in \mathcal{O}_0^{\text{qc}}$ mapping into K_{int} is open by Theorem 3.4.

Defining $\alpha(t) = \mathcal{A}(h) \circ f_t \cdot f'_t$ and denoting t -differentiation with a dot we then have that

$$\lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{A}(h \circ f_t) - \mathcal{A}(h \circ f_0)) = \dot{\alpha}(t) + \phi.$$

So it is enough to show that

$$(3.12) \quad \left\| \frac{1}{t} (\mathcal{A}(h \circ f_t) - \mathcal{A}(h \circ f_0)) - (\dot{\alpha}(t) + \phi) \right\| = \left\| \frac{1}{t} (\alpha(t) - \alpha(0) - t\dot{\alpha}(0)) \right\| \rightarrow 0$$

as $t \rightarrow 0$. For any fixed z (recall that $\alpha(t)$ is also a function of z) we have

$$\alpha(t) - \alpha(0) - t\dot{\alpha}(0) = \int_0^t \ddot{\alpha}(s)(t-s) ds.$$

We claim that there is a constant C_0 such that $\|\ddot{\alpha}\| < C_0$ for all t in some neighborhood of 0. Assuming for the moment that this is true, for $|s| < |t| < C$ we set $t = e^{i\theta}u$ and $s = e^{i\theta}v$, and integrating along a ray, we have

$$\begin{aligned} \|\alpha(t) - \alpha(0) - t\dot{\alpha}(0)\|^2 &= \left\| \int_0^t \ddot{\alpha}(s)(t-s) ds \right\|^2 \\ &= \iint_{\mathbb{D}} \left| \int_0^t \ddot{\alpha}(s)(t-s) ds \right|^2 dA \\ &\leq \iint_{\mathbb{D}} \left(\int_0^u |\ddot{\alpha}(e^{i\theta}v)|(u-v) dv \right)^2 dA \\ &\leq \iint_{\mathbb{D}} \int_0^u u |\ddot{\alpha}(e^{i\theta}v)|^2 (u-v)^2 dv dA \\ &\leq C \iint_{\mathbb{D}} \int_0^u |\ddot{\alpha}(e^{i\theta}v)|^2 (u-v)^2 dv dA \end{aligned}$$

where we have used Jensen's inequality and the assumption that $u < C$. Therefore Fubini's theorem and the assumption that $v < u < |t|$ yield

$$\begin{aligned} \|\alpha(t) - \alpha(0) - t\dot{\alpha}(0)\|^2 &\leq 4C|t|^2 \int_0^{|t|} \left(\iint_{\mathbb{D}} |\ddot{\alpha}(s)|^2 dA \right) d|s| \\ &\leq C_1|t|^3. \end{aligned}$$

Fubini's theorem can be applied since the second to last integral converges by the final inequality. This would prove (3.12). Thus the proof reduces to establishing a bound on $\|\ddot{\alpha}\|$ which is uniform in t in some neighborhood of 0.

By [24, equation 3.2],

$$\begin{aligned}
 (3.13) \quad \ddot{\alpha}(t) &= \mathcal{A}(h)'' \circ f_t \cdot f_t' \cdot \dot{f}_t^2 + \mathcal{A}(h)' \circ f_t \cdot f_t' \cdot \ddot{f}_t \\
 &\quad + 2\mathcal{A}(h)' \circ f_t \cdot \dot{f}_t \cdot \dot{f}_t' + \mathcal{A}(h) \circ f_t \cdot \ddot{f}_t' \\
 &= I + II + III + IV
 \end{aligned}$$

where

$$\mathcal{A}(h)' = \frac{h'''}{h'} - \frac{h''^2}{h'^2}$$

and

$$\mathcal{A}(h)'' = \frac{h''''}{h'} - 3\frac{h''''h''}{h'^2} - \frac{h''^3}{h'^3}.$$

We will uniformly bound all the terms on the right side of (3.13) in the $A_1^2(\mathbb{D})$ norm. For all $t \in N$ we have $\overline{f_t(\mathbb{D})} \subset K$ and h is holomorphic on an open set containing the compact set K , and $h' \neq 0$ since h is one-to-one on A . Thus there is a uniform bound for $\mathcal{A}(h)$, $\mathcal{A}(h)'$ and $\mathcal{A}(h)''$ on $f_t(\mathbb{D})$. So by a change of variables, there is an M such that

$$(3.14) \quad \|\mathcal{A}(h) \circ f_t \cdot f_t'\| = \left(\iint_{f_t(\mathbb{D})} |\mathcal{A}(h)|^2 dA \right)^{1/2} \leq M.$$

Similarly there are M' and M'' such that

$$(3.15) \quad \|\mathcal{A}(h)' \circ f_t \cdot f_t'\| \leq M' \quad \text{and} \quad \|\mathcal{A}(h)'' \circ f_t \cdot f_t'\| \leq M''.$$

Since $\overline{f_t(\mathbb{D})}$ is contained in the compact set K , $|f_t(z)|$ is bounded by a constant C which is independent of t . By applying Cauchy estimates in the variable t on a curve $|t| = r_2$, we see that for $0 < r_1 < r_2$ and $|t| \leq r_1$,

$$|\dot{f}_t(z)| \leq \frac{r_2}{(r_1 - r_2)^2} \sup_{|s|=r_2} |f_s(z)|$$

and thus we can find a constant C' such that $|\dot{f}_t(z)| \leq C'$ for $|t| \leq r_1$. Similarly, there is a C'' such that $|\ddot{f}_t(z)| \leq C''$ for all $z \in \mathbb{D}$ and $|t| \leq r_1$. Combining with (3.15), we have that $\|I\|$ and $\|II\|$ are uniformly bounded on $|t| \leq r_1$.

Next, observe that $\|\mathcal{A}(h)' \circ f_t\|_\infty \leq D$ and $\|\mathcal{A}(h) \circ f_t\|_\infty \leq D'$ for some constants D and D' which are independent of t , since $f_t(\mathbb{D})$ is contained inside a compact set in the interior of the domain of h , and h is holomorphic and one-to-one. Therefore, to get a uniform bound on $\|\ddot{\alpha}\|$ we only need to show that $\|\dot{f}_t'\|$ and $\|\ddot{f}_t'\|$ are bounded by some constant which is independent of t on a neighborhood of 0.

A simple computation yields

$$\dot{f}_t'(z) = \frac{\dot{q}(t)}{q(t)} f_t'(z) + \left(\int_0^z \phi(w) dw \right) f_t'(z).$$

Since $q(t)$ is holomorphic and non-zero, \dot{q}/q is uniformly bounded on a neighborhood of 0. Furthermore,

$$\iint_{\mathbb{D}} |f_t'|^2 dA = \text{Area}(f_t(\mathbb{D}))$$

which is uniformly bounded since $f_t(\mathbb{D})$ is contained in a fixed compact set. Since $\psi(z) = \int_0^z \phi(w)dw$ is in the Dirichlet space, we can apply Lemma 3.8 with $\beta = 2$, which proves that $\|\dot{f}_t'\|$ is uniformly bounded for t in some neighborhood of 0. We further compute that

$$\ddot{f}_t'(z) = \frac{\ddot{q}(t)}{q(t)} f_t'(z) + 2 \frac{\dot{q}(t)}{q(t)} \left(\int_0^z \phi(w)dw \right) f_t'(z) + \left(\int_0^z \phi(w)dw \right)^2 f_t'(z),$$

so the same reasoning (this time using Lemma 3.8 with $\beta = 2$ and $\beta = 4$) yields a uniform bound for $\|\ddot{f}_t'\|$. This completes the proof. \square

3.2. Complex Hilbert manifold structure on $\mathcal{O}_0^{\text{qc}}(\Sigma)$. The idea behind the complex Hilbert space structure is as follows. Any element (f_1, \dots, f_n) of $\mathcal{O}_0^{\text{qc}}(\Sigma)$ maps n closed discs onto closed sets containing the punctures. We choose charts ζ_i , $i = 1, \dots, n$, which map non-overlapping open neighborhoods of the closed discs into \mathbb{C} . The maps $\zeta_i \circ f_i$ are in $\mathcal{O}_0^{\text{qc}}$, which is an open subset of a Hilbert space. By Theorem 3.4 the components g_i of an element g nearby to f will also have images in the domains of the charts ζ_i . Thus we can model $\mathcal{O}_0^{\text{qc}}(\Sigma)$ locally by $\mathcal{O}_0^{\text{qc}} \times \dots \times \mathcal{O}_0^{\text{qc}}$. Theorem 3.9 will ensure that the transition functions of the charts are biholomorphisms.

We now turn to the proofs, beginning with the topology on $\mathcal{O}_0^{\text{qc}}(\Sigma)$. Before defining a topological basis we need some notation.

Definition 3.11. For any n -chart $(\zeta, E) = (\zeta_1, E_1, \dots, \zeta_n, E_n)$ (see Definition 3.2), we say that an n -tuple $U = (U_1, \dots, U_n) \subset \mathcal{O}_0^{\text{qc}} \times \dots \times \mathcal{O}_0^{\text{qc}}$, with U_i open in $\mathcal{O}_0^{\text{qc}}$, is compatible with (ζ, E) if $\overline{f(\mathbb{D})} \subset \zeta_i(E_i)$ for all $f \in U_i$.

For any n -chart (ζ, E) and compatible open subset U of $\mathcal{O}_0^{\text{qc}} \times \dots \times \mathcal{O}_0^{\text{qc}}$ let

$$(3.16) \quad \begin{aligned} V_{\zeta, E, U} &= \{g \in \mathcal{O}_0^{\text{qc}}(\Sigma) : \zeta_i \circ g_i \in U_i, \quad i = 1, \dots, n\} \\ &= \{(\zeta_1^{-1} \circ h_1, \dots, \zeta_n^{-1} \circ h_n) : h_i \in U_i, \quad i = 1, \dots, n\}. \end{aligned}$$

Definition 3.12 (base a for topology on $\mathcal{O}_0^{\text{qc}}(\Sigma)$). Let

$$\mathcal{V} = \{V_{\zeta, E, U} : (\zeta, E) \text{ an } n\text{-chart, } U \text{ compatible with } (\zeta, E)\}.$$

Theorem 3.13. *The set \mathcal{V} is the base for a topology on $\mathcal{O}_0^{\text{qc}}(\Sigma)$. This topology is Hausdorff and second countable.*

Proof. We first establish that \mathcal{V} is a base. For any element f of $\mathcal{O}_0^{\text{qc}}(\Sigma^P)$, since $\overline{f_i(\mathbb{D})}$ is compact for all i , there is an n -chart (ζ, E) such that $\overline{f_i(\mathbb{D})} \subset E_i$ for each i . By Theorem 3.4 there is a $U = (U_1, \dots, U_n)$ compatible with (ζ, E) . Thus \mathcal{V} covers $\mathcal{O}_0^{\text{qc}}(\Sigma^P)$.

Now let $V_{\zeta, E, U}$ and $V_{\zeta', E', U'}$ be two elements of \mathcal{V} containing a point $f \in \mathcal{O}_0^{\text{qc}}(\Sigma^P)$. Define E'' by $E_i'' = E_i \cap E_i'$. For each i choose a compact set κ_i such that $\overline{f_i(\mathbb{D})} \subseteq \kappa_i \subseteq E_i''$. Let $K_i = \zeta_i(\kappa_i)$, $K_i' = \zeta_i'(\kappa_i)$,

$$W_i = \{\phi \in \mathcal{O}_0^{\text{qc}} : \overline{\phi(\mathbb{D})} \subseteq K_i^{\text{int}}\}$$

and

$$W_i' = \{\phi \in \mathcal{O}_0^{\text{qc}} : \overline{\phi(\mathbb{D})} \subseteq K_i'^{\text{int}}\}$$

where K_i^{int} and $K_i'^{\text{int}}$ are the interiors of K_i and K_i' respectively. By Theorem 3.4 W_i and W_i' are open, and by Theorem 3.9 the map $\phi \mapsto \zeta_i' \circ \zeta_i^{-1} \circ \phi$ is a biholomorphism from W_i onto W_i' . So the set

$$U_i'' = U_i \cap \left(\zeta_i \circ \zeta_i'^{-1} (W_i' \cap U_i') \right) \subseteq U_i \cap W_i$$

is an open subset of $\mathcal{O}_0^{\text{qc}}$ (by $\zeta_i'^{-1}(W_i' \cap U_i')$ we mean the set of $\zeta_i'^{-1} \circ \phi$ for $\phi \in W_i' \cap U_i'$). Setting $\zeta_i'' = \zeta|_{E_i''}$ we have that $f \in V_{\zeta'', E'', U''} \subseteq V_{\zeta, E, U} \cap V_{\zeta', E', U'}$ by construction. Thus \mathcal{V} is a base.

To show that the topology generated by \mathcal{V} is Hausdorff, let $f, g \in \mathcal{O}_0^{\text{qc}}(\Sigma^P)$. Choose open, simply connected sets E_i and F_i , $i = 1, \dots, n$ such that $\overline{f_i(\mathbb{D})} \subset E_i$ and $\overline{g_i(\mathbb{D})} \subset F_i$ and $E_i \cap E_j = F_i \cap F_j = \emptyset$ whenever $i \neq j$. For each i let $\zeta_i : E_i \cup F_i \rightarrow \mathbb{C}$ be a biholomorphism taking p_i to 0. Thus $\zeta_i|_{E_i}$ defines an n -chart (ζ, E) , and similarly for $\zeta_i|_{F_i}$. (The collection $\zeta_i|_{E_i \cup F_i}$ does not necessarily form an n -chart, but this is inconsequential).

Since $\mathcal{O}_0^{\text{qc}}$ is a Hilbert space, it is Hausdorff, so for all i there are open sets U_i and W_i such that $\zeta_i \circ f_i \in U_i$, $\zeta_i \circ g_i \in W_i$, and $U_i \cap W_i = \emptyset$. By Theorem 3.4, by shrinking U_i and W_i if necessary, we can assume that $\overline{h_i(\mathbb{D})} \subset \zeta_i(E_i)$ for all $h_i \in U_i$ and $\overline{h_i(\mathbb{D})} \subset \zeta_i(F_i)$ for all $h_i \in W_i$. That is, U is compatible with (ζ, E) and W is compatible with (ζ, F) . Furthermore $f \in V_{\zeta, E, U}$, $g \in V_{\zeta, F, W}$ and $V_{\zeta, E, U} \cap V_{\zeta, F, W} = \emptyset$ by construction. Thus $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is Hausdorff with the topology defined by \mathcal{V} .

To see that $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is second countable, we proceed as follows. First observe that Σ is second countable by Rado's Theorem (see for example [17]). Thus it has a countable basis \mathfrak{B} of open sets. Let $\mathfrak{B}^n = \{(B_1, \dots, B_n)\}$ where each B_i (1) is a finite union of elements of \mathfrak{B} and (2) contains p_i . Clearly \mathfrak{B}^n is countable. Consider the set of n -tuples $C = (C_1, \dots, C_n)$ such that (1) $(C_1, \dots, C_n) \in \mathfrak{B}^n$ and (2) $C_i \cap C_j$ is empty whenever $i \neq j$. Since this is a subset of \mathfrak{B}^n , it is countable. Furthermore, for each (C_1, \dots, C_n) , we can fix a chart $\zeta_i : C_i \rightarrow \mathbb{C}$. Let \mathfrak{C} be the collection of n -charts $\{(\zeta_1, C_1, \dots, \zeta_n, C_n)\}$ where ζ_i and C_i are as above.

Next, since $\mathcal{O}_0^{\text{qc}}$ is a Hilbert space (and hence a separable metric space), it has a countable basis of open sets \mathfrak{D} . We define a countable basis for the topology of $\mathcal{O}_0^{\text{qc}}(\Sigma)$ as follows:

$$\mathcal{V}' = \{V_{(\zeta, C, W)} : (\zeta, C) \in \mathfrak{C}, W \text{ compatible with } (\zeta, C), W_i \in \mathfrak{D}, i = 1, \dots, n\}.$$

Each $V' \in \mathcal{V}'$ is open by Theorem 3.4. Furthermore \mathcal{V}' is countable since \mathfrak{C} and \mathfrak{D} are countable. We need to show that \mathcal{V}' is a base for the topology of $\mathcal{O}_0^{\text{qc}}(\Sigma)$. Clearly $\mathcal{V}' \subset \mathcal{V}$. Thus it is enough to show that for every $f = (f_1, \dots, f_n) \in \mathcal{O}_0^{\text{qc}}(\Sigma)$ and $V \in \mathcal{V}$ containing f , there is a $V' \in \mathcal{V}'$ such that $f \in V' \subset V$.

Let $V_{\zeta, E, U} \in \mathcal{V}$ contain f . We claim that there is an n -chart $(\eta, C) \in \mathfrak{C}$ such that $\overline{f_i(\mathbb{D})} \subset C_i \subset E_i$ for all i . To see this, fix i and observe that since \mathfrak{B} is a base for Σ , for each point $x \in \overline{f_i(\mathbb{D})}$ there is an open set $B_{i,x} \in \mathfrak{B}$ such that $x \in B_{i,x} \subset E_i$. The set $\{B_{i,x}\}_{x \in \overline{f_i(\mathbb{D})}}$ is a cover of $\overline{f_i(\mathbb{D})}$; since it is compact there is a finite subcover say $\{B_{i,\alpha}\}$. Set $C_i = \cup_{\alpha} B_{i,\alpha}$ and perform this procedure for each $i = 1, \dots, n$. By construction the C_i are non-overlapping and $C = (C_1, \dots, C_n) \in \mathfrak{B}^n$. It follows that $(\eta, C) = (\eta_1, C_1, \dots, \eta_n, C_n) \in \mathfrak{C}$ where η_i are the charts corresponding to C_i . This proves the claim.

Since \mathfrak{D} is a basis of $\mathcal{O}_0^{\text{qc}}$, by Theorems 3.4 and 3.9 (using an argument similar to the one earlier in the proof), for each i there is a $W_i \in \mathfrak{D}$ satisfying $\eta_i \circ f_i \in W_i \subset \eta_i \circ \zeta_i^{-1}(U_i)$. If $g \in V'_{\eta, C, W}$ then $g_i = \eta_i^{-1} \circ h_i$ for some $h_i \in W_i$ for all $i = 1, \dots, n$ by (3.16). But $h_i \in \eta_i \circ \zeta_i^{-1}(U_i)$, so $g_i \in \zeta_i^{-1}(U_i)$ and hence $g \in V_{\zeta, E, U}$ by (3.16). Thus $V'_{\eta, C, W} \subset V_{\zeta, E, U}$ which completes the proof. \square

Remark 3.14. In particular, $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is separable since it is second countable and Hausdorff.

We make one final simple but useful observation regarding the base \mathcal{V} .

For a Riemann surface Σ denote by $\mathcal{V}(\Sigma)$ the base for $\mathcal{O}_0^{\text{qc}}(\Sigma)$ given in Definition 3.12. For a biholomorphism $\rho : \Sigma \rightarrow \Sigma_1$ of Riemann surfaces Σ and Σ_1 , and for any $V \in \mathcal{V}(\Sigma)$, let

$$\rho(V) = \{\rho \circ \phi : \phi \in V\}$$

and

$$\rho(\mathcal{V}(\Sigma)) = \{\rho(V) : V \in \mathcal{V}\}.$$

Theorem 3.15. *If $\rho : \Sigma \rightarrow \Sigma_1$ is a biholomorphism between punctured Riemann surfaces Σ and Σ_1 then $\rho(\mathcal{V}(\Sigma)) = \mathcal{V}(\Sigma_1)$.*

Proof. It is an immediate consequence of Definition 3.12 and Theorem 3.9 that $\rho(\mathcal{V}(\Sigma)) \subseteq \mathcal{V}(\Sigma_1)$. Similarly $\rho^{-1}(\mathcal{V}(\Sigma_1)) \subseteq \mathcal{V}(\Sigma)$. Since $\rho(\rho^{-1}(\mathcal{V}(\Sigma_1))) = \mathcal{V}(\Sigma_1)$ and $\rho^{-1}(\rho(\mathcal{V}(\Sigma))) = \mathcal{V}(\Sigma)$ the result follows. \square

Definition 3.16 (standard charts on $\mathcal{O}_0^{\text{qc}}(\Sigma)$). Let (ζ, E) be an n -chart on Σ and let $\kappa_i \subset E_i$ be compact sets containing p_i . Let $K_i = \zeta_i(\kappa_i)$. Let $U_i = \{\psi \in \mathcal{O}_0^{\text{qc}} : \overline{\psi(\mathbb{D})} \subset \text{interior}(K_i)\}$. Each U_i is open by Theorem 3.4 and $U = (U_1, \dots, U_n)$ is compatible with (ζ, E) so we have $V_{\zeta, E, U} \in \mathcal{V}$. A standard chart on $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is a map

$$\begin{aligned} T : V_{\zeta, E, U} &\longrightarrow \mathcal{O}_0^{\text{qc}} \times \dots \times \mathcal{O}_0^{\text{qc}} \\ (f_1, \dots, f_n) &\longmapsto (\zeta_1 \circ f_1, \dots, \zeta_n \circ f_n). \end{aligned}$$

Remark 3.17. To obtain a chart into a Hilbert space, one simply composes with χ as defined by (2.2). Abusing notation somewhat and defining χ^n by

$$\begin{aligned} \chi^n \circ T : V_{\zeta, E, U} &\longrightarrow \bigoplus_{i=1}^n A_1^2(\mathbb{D}) \oplus \mathbb{C} \\ (f_1, \dots, f_n) &\longmapsto (\chi \circ \zeta_1 \circ f_1, \dots, \chi \circ \zeta_n \circ f_n) \end{aligned}$$

we obtain a chart into $\bigoplus_{i=1}^n A_1^2(\mathbb{D}) \oplus \mathbb{C}$. Since $\chi(\mathcal{O}_0^{\text{qc}})$ is an open subset of $A_1^2(\mathbb{D}) \oplus \mathbb{C}$ by Theorem 2.3, and χ defines the complex structure $\mathcal{O}_0^{\text{qc}}$, we may treat T as a chart with the understanding that the true charts are obtained by composing with χ^n .

Theorem 3.18. *Let Σ be a punctured Riemann surface of type (g, n) . With the atlas consisting of the standard charts of Definition 3.16, $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is a complex Hilbert manifold, locally biholomorphic to $\mathcal{O}_0^{\text{qc}} \times \dots \times \mathcal{O}_0^{\text{qc}}$.*

Proof. We have already shown that $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is Hausdorff and separable (in fact second countable). So we need only show that the charts above form an atlas of homeomorphisms with biholomorphic transition functions.

Let $V = V_{\zeta, E, U}$ and $V' = V_{\zeta', E', U'}$ where U and U' are determined by compact sets κ_i and κ'_i respectively, as in Definition 3.16. With the topology from the basis \mathcal{V} of Definition 3.12 the charts are automatically homeomorphisms. It suffices to show that for two standard charts $T : V \rightarrow \mathcal{O}_0^{\text{qc}} \times \dots \times \mathcal{O}_0^{\text{qc}}$ and $T' : V' \rightarrow \mathcal{O}_0^{\text{qc}} \times \dots \times \mathcal{O}_0^{\text{qc}}$ the overlap maps $T \circ T'^{-1}$ and $T' \circ T^{-1}$ are holomorphic.

Assume that $V \cap V'$ is non-empty. For $(\psi_1, \dots, \psi_n) \in T'(V \cap V')$

$$T \circ T'^{-1}(\psi_1, \dots, \psi_n) = (\zeta_1 \circ \zeta_1'^{-1} \circ \psi_1, \dots, \zeta_n \circ \zeta_n'^{-1} \circ \psi_n).$$

The maps $\psi_i \mapsto \zeta_i \circ \zeta_i'^{-1} \circ \psi_i$ are holomorphic maps of $\zeta_i'(V_i \cap V_i')$ by Theorem 3.9 with $A = \zeta_i'(E_i \cap E_i')$, $U = \zeta_i'(\zeta_i^{-1}(U_i) \cap \zeta_i'^{-1}(U_i')) = \zeta_i' \circ \zeta_i^{-1}(U_i) \cap U_i'$, $K = \zeta_i' \circ \zeta_i^{-1}(K_i) \cap K_i'$ and $h = \zeta_i \circ \zeta_i'^{-1}$. Similarly $T' \circ T^{-1}$ is holomorphic. \square

Remark 3.19 (chart simplification). Now that this theorem is proven, we can simplify the definition of the charts. For an n -chart (ζ, E) , if we let $U_i = \{f \in \mathcal{O}_0^{\text{qc}} : \overline{f(\mathbb{D})} \subset \zeta_i(E_i)\}$, then the charts T are defined on $V_{\zeta, E, U}$. It is easy to show that T is a biholomorphism on $V_{\zeta, E, U}$, since any $f \in V_{\zeta, E, U}$ is contained in some $V_{\zeta, E, W} \subset V_{\zeta, E, U}$ which satisfies Definition 3.16, and thus T is a biholomorphism on $V_{\zeta, E, W}$ by Theorem 3.18.

Remark 3.20 (standard charts on $\mathcal{O}^{\text{qc}}(\Sigma)$). A standard chart on $\mathcal{O}^{\text{qc}}(\Sigma)$ is defined in the same way as Definition 3.16 and its preamble, by replacing $\mathcal{O}_0^{\text{qc}}$ with \mathcal{O}^{qc} everywhere. Furthermore with this atlas $\mathcal{O}^{\text{qc}}(\Sigma)$ is a complex Banach manifold [24].

Finally, we show that the inclusion map $I : \mathcal{O}_0^{\text{qc}} \rightarrow \mathcal{O}^{\text{qc}}$ is holomorphic.

Theorem 3.21. *The complex manifold $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is holomorphically contained in $\mathcal{O}^{\text{qc}}(\Sigma)$ in the sense that the inclusion map $I : \mathcal{O}_0^{\text{qc}}(\Sigma) \rightarrow \mathcal{O}^{\text{qc}}(\Sigma)$ is holomorphic.*

Proof. This follows directly from the construction of the charts on $\mathcal{O}^{\text{qc}}(\Sigma)$. Let $T : V \rightarrow \mathcal{O}^{\text{qc}} \times \cdots \times \mathcal{O}^{\text{qc}}$ be a standard chart on $\mathcal{O}^{\text{qc}}(\Sigma)$ as specified in Remark 3.20. Let $U = T(V)$ and $U_0 = U \cap \mathcal{O}_0^{\text{qc}} \times \cdots \times \mathcal{O}_0^{\text{qc}}$. Let $V_0 = T^{-1}(U_0)$. The map $T|_{V_0}$ is a chart on $V_0 \subseteq \mathcal{O}_0^{\text{qc}}(\Sigma)$, so it is holomorphic in the refined setting. Since the inclusion map $\iota : U_0 \rightarrow U$ is holomorphic by Theorem 2.3, the inclusion map $I = T^{-1} \circ \iota \circ (T|_{V_0})$ is holomorphic on V_0 . Since $\mathcal{O}_0^{\text{qc}}(\Sigma)$ is covered by charts of this form, I is holomorphic. \square

4. THE RIGGED TEICHMÜLLER SPACE IS A HILBERT MANIFOLD

In [23], two of the authors proved that the Teichmüller space of a bordered surface is (up to a quotient by a discrete group) the same as a certain rigged Teichmüller space whose corresponding rigged moduli space appears naturally in two-dimensional conformal field theory [9, 15, 26]. We will use this fact to define a Hilbert manifold structure on the refined Teichmüller space in Section 5.

First we must define an atlas on rigged Teichmüller space, and this is the main task of the current section. We will achieve this by using universality of the universal Teichmüller curve together with a variational technique called *Schiffer variation* as adapted to the quasiconformal Teichmüller setting by Gardiner [10] and Nag [20, 21]. This overall approach was first developed in the thesis of the first author [22] for the case of analytic riggings.

4.1. Definition of rigged Teichmüller space. We first recall the definition of the usual Teichmüller space. The reader is referred to Section 2.2 for terminology regarding Riemann surfaces.

Definition 4.1. Fix a Riemann surface X (of any topological type). Let

$$T(X) = \{(X, f, X_1)\} / \sim$$

where

- (1) X_1 is a Riemann surface of the same topological type as X .
- (2) $f : X \rightarrow X_1$ is a quasiconformal homeomorphism (the *marking map*).
- (3) the equivalence relation (\sim) is defined by $(X, f_1, X_1) \sim (X, f_2, X_2)$ if and only if there exists a biholomorphism $\sigma : X_1 \rightarrow X_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity *rel boundary*.

The term *rel boundary* means that the homotopy is the identity on the boundary throughout the homotopy.

It is a standard fact of Teichmüller theory (see for example [21]) that if X is a punctured surface of type (g, n) then $T(X)$ is a complex manifold of dimension $3g - 3 + n$, and if X is a bordered surface of type (g, n) then $T(X)$ is an infinite-dimensional complex Banach manifold.

Using the set $\mathcal{O}_0^{\text{qc}}(\Sigma)$ we now define the (*refined*) *rigged Teichmüller space*, denoted by $\tilde{T}_0(\Sigma)$.

Definition 4.2. Fix a punctured Riemann surface of type (g, n) . Let

$$\tilde{T}_0(\Sigma) = \{(\Sigma, f, \Sigma_1, \phi)\} / \sim$$

where

- (1) Σ_1 is a punctured Riemann surface of type (g, n)
- (2) $f : \Sigma \rightarrow \Sigma_1$ is a quasiconformal homeomorphism
- (3) $\phi \in \mathcal{O}_0^{\text{qc}}(\Sigma_1)$.
- (4) Two quadruples are said to be equivalent, denoted by $(\Sigma, f_1, \Sigma_1, \phi_1) \sim (\Sigma, f_2, \Sigma_2, \phi_2)$, if and only if there exists a biholomorphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel boundary and $\phi_2 = \sigma \circ \phi_1$.

The equivalence class of $(\Sigma, f_1, \Sigma_1, \phi_1)$ will be denoted $[\Sigma, f_1, \Sigma_1, \phi_1]$

Condition (2) can be stated in two alternate ways. One is to require that f maps the compactification of Σ into the compactification of Σ_1 , and takes the punctures of Σ to the punctures of Σ_1 (now thought of as marked points). The other is to say simply that f is a quasiconformal map between Σ and Σ_1 . Since f is quasiconformal its extension to the compactification will take punctures to punctures. Thus condition (2) does not explicitly mention the punctures.

In [23], two of the authors defined a rigged Teichmüller space $\tilde{T}(\Sigma)$ obtained by replacing $\mathcal{O}_0^{\text{qc}}(\Sigma_1)$ with $\mathcal{O}^{\text{qc}}(\Sigma_1)$ in the above definition. It was demonstrated in [23] that $\tilde{T}(\Sigma)$ has a complex Banach manifold structure, which comes from the fact that it is a quotient of the Teichmüller space of a bordered surface by a properly discontinuous, fixed-point free group of biholomorphisms. In [25] we demonstrated that it is fibred over $T(\Sigma)$, where the fiber over a point $[\Sigma, f_1, \Sigma_1]$ is biholomorphic to $\mathcal{O}^{\text{qc}}(\Sigma_1)$. Furthermore, the complex structure of $\mathcal{O}^{\text{qc}}(\Sigma_1)$ is compatible with the complex structure that the fibres inherit from $\tilde{T}(\Sigma)$.

This notion of a *rigged Teichmüller space* was first defined, in the case of analytic riggings, by one of the authors in [22], and it was used to obtain a complex Banach manifold structure on the analytically rigged moduli space. However, in this case the connection to the complex structure of the infinite-dimensional Teichmüller space of bordered surfaces can not be made.

From now on, any punctured Riemann surface is assumed to satisfy $2g + 2 - n > 0$. We would now like to demonstrate that $\tilde{T}_0(\Sigma)$ has a natural complex Hilbert manifold structure which arises from $\mathcal{O}_0^{\text{qc}}(\Sigma)$, and that this also passes to the rigged Riemann moduli space. In Section 5, we will use it to construct a complex Hilbert manifold structure on a refined Teichmüller space of a bordered surface. To accomplish these tasks we use a natural coordinate system developed in [22, 25], which is based on Gardiner-Schiffer variation and the complex structure on $\mathcal{O}^{\text{qc}}(\Sigma)$. We will refine these coordinates to $\tilde{T}_0(\Sigma)$.

We end this section with a basic result concerning the above definition. Since Σ satisfies $2g + 2 - n > 0$ we have the following well known theorem [21].

Theorem 4.3. *If $\sigma : \Sigma \rightarrow \Sigma$ is a biholomorphism that is homotopic to the identity then σ is the identity.*

Corollary 4.4. *If $[\Sigma, f_1, \Sigma_1, \phi_1] = [\Sigma, f_1, \Sigma_1, \phi_2] \in \tilde{T}_0(\Sigma)$ then $\phi_1 = \phi_2$.*

4.2. Marked families. In this section we collect some standard definitions and facts about marked holomorphic families of Riemann surfaces and the universality of the Teichmüller curve. These will play a key role in the construction of an atlas on rigged Teichmüller space. A full treatment appears in [6], and also in the books [17, 21].

Definition 4.5. A holomorphic family of complex manifolds is a pair of connected complex manifolds (E, B) together with a surjective holomorphic map $\pi : E \rightarrow B$ such that (1) π is topologically a locally trivial fibre bundle, and (2) π is a split submersion (that is, the derivative is a surjective map whose kernel is a direct summand).

Definition 4.6. A *morphism of holomorphic families* from (E', B') and (E, B) is a pair of holomorphic maps (α, β) with $\alpha : B' \rightarrow B$ and $\beta : E' \rightarrow E$ such that

$$\begin{array}{ccc} E' & \xrightarrow{\beta} & E \\ \pi' \downarrow & & \downarrow \pi \\ B' & \xrightarrow{\alpha} & B \end{array}$$

commutes, and for each fixed $t \in B'$, the restriction of β to the fibre $\pi'^{-1}(t)$ is a biholomorphism onto $\pi^{-1}(\alpha(t))$.

Throughout, (E, B) will be a holomorphic family of Riemann surfaces; that is, each fibre $\pi^{-1}(t)$ is a Riemann surface. Moreover, since our trivialization will always be global we specialize the standard definitions (see [6]) to this case in what follows.

Let Σ be a punctured Riemann surface of type (g, n) . This fixed surface Σ will serve as a model of the fibre.

Definition 4.7.

- (1) A *global trivialization* of (E, B) is a homeomorphism $\theta : B \times \Sigma \rightarrow E$ such that $\pi(\theta(t, x)) = t$ for all $(t, x) \in B \times \Sigma$.
- (2) A global trivialization θ is a *strong trivialization* if for fixed $x \in \Sigma$, $t \mapsto \theta(t, x)$ is holomorphic, and for each $t \in B$, $x \mapsto \theta(t, x)$ is a quasiconformal map from Σ onto $\pi^{-1}(t)$.
- (3) $\theta : B \times \Sigma \rightarrow E$ and $\theta' : B \times \Sigma \rightarrow E$ are *compatible* if and only if $\theta'(t, x) = \theta(t, \phi(t, x))$ where for each fixed t , $\phi(t, x) : \Sigma \rightarrow \Sigma$ is a quasiconformal homeomorphism that is homotopic to the identity rel boundary.
- (4) A *marking* \mathcal{M} for $\pi : E \rightarrow B$ is an equivalence class of compatible strong trivializations.
- (5) A *marked family of Riemann surfaces* is a holomorphic family of Riemann surfaces with a specified marking.

Remark 4.8. Let θ and θ' be compatible strong trivializations. For each fixed $t \in B$, $[\Sigma, \theta(t, \cdot), \pi^{-1}(t)] = [\Sigma, \theta'(t, \cdot), \pi^{-1}(t)]$ in $T(\Sigma)$ (see Definition 4.1). So a marking specifies a Teichmüller equivalence class for each t .

We now define the equivalence of marked families.

Definition 4.9. A *morphism of marked families* from $\pi' : E' \rightarrow B'$ to $\pi : E \rightarrow B$ is a pair of holomorphic maps (α, β) with $\beta : E' \rightarrow E$ and $\alpha : B' \rightarrow B$ such that

- (1) (α, β) is a morphism of holomorphic families, and
- (2) the markings $B' \times \Sigma \rightarrow E$ given by $\beta(\theta'(t, x))$ and $\theta(\alpha(t), x)$ are compatible.

The second condition says that (α, β) preserves the marking.

Remark 4.10 (relation to Teichmüller equivalence). Define $E = \{(s, Y_s)\}_{s \in B}$ and $E' = \{(t, X_t)\}_{t \in B'}$ to be marked families of Riemann surfaces with markings $\theta(s, x) = (s, g_s(x))$ and $\theta'(t, x) = (t, f_t(x))$ respectively. Say (α, β) is a morphism of marked families, and define σ_t by $\beta(t, y) = (\alpha(t), \sigma_t(y))$. Then $\beta(\theta'(t, x)) = (\alpha(t), \sigma_t(f_t(x)))$ and $\theta(\alpha(t), x) = (\alpha(t), g_{\alpha(t)}(x))$. The condition that (α, β) is a morphism of marked families is simply that $\sigma_t \circ f_t$ is homotopic rel boundary to $g_{\alpha(t)}$. That is, when $s = \alpha(t)$, $[\Sigma, f_t, X_t] = [\Sigma, g_s, Y_s]$ via the biholomorphism $\sigma_t : X_t \rightarrow Y_s$.

The universal Teichmüller curve, denoted by $\pi_T : \mathcal{T}(\Sigma) \rightarrow T(\Sigma)$, is a marked holomorphic family of Riemann surfaces with fibre model Σ . The following universal property of $\mathcal{T}(\Sigma)$ (see [6, 17, 21]) is all that we need for our purposes.

Theorem 4.11 (Universality of the Teichmüller curve). *Let $\pi : E \rightarrow B$ be a marked holomorphic family of Riemann surfaces with fibre model Σ of type (g, n) with $2g - 2 + n > 0$, and trivialization θ . Then there exists a unique map (α, β) of marked families from $\pi : E \rightarrow B$ to $\pi_T : \mathcal{T}(\Sigma) \rightarrow T(\Sigma)$. Moreover, the canonical “classifying” map $\alpha : B \rightarrow T(\Sigma)$ is given by $\alpha(t) = [\Sigma, \theta(t, \cdot), \pi^{-1}(t)]$.*

4.3. Schiffer variation. The use of Schiffer variation to construct holomorphic coordinates for Teichmüller space by using quasiconformal deformations is due to Gardiner [10] and Nag [20, 21]. We review the construction in some detail, as it will be used in a crucial way.

Let $B_R = \{z \in \mathbb{C} : |z| < R\}$, and for $r < R$ let $\mathbb{A}(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$ as before. Choose r and R such that $0 < r < 1 < R$. Let Σ be a (possibly punctured) Riemann surface and $\xi : U \rightarrow \mathbb{C}$ be local holomorphic coordinate on an open connected set $U \subset \Sigma$ such that $B_R \subset \text{Image}(\xi)$. Let $D = \xi^{-1}(\mathbb{D})$, which we call a parametric disk.

Define $v^\epsilon : \mathbb{A}(r, R) \rightarrow \mathbb{C}$ by $v^\epsilon(z) = z + \epsilon/z$ where $\epsilon \in \mathbb{C}$. For $|\epsilon|$ sufficiently small v is a biholomorphism onto its image. Let D^ϵ be the interior of the analytic Jordan curve $v^\epsilon(\partial\mathbb{D})$. We regard $\overline{D^\epsilon}$ as a bordered Riemann surface (with the standard complex structure inherited from \mathbb{C}) with analytic boundary parametrization given by $v^\epsilon : S^1 \rightarrow \partial D^\epsilon$. We also have the Riemann surface $\Sigma \setminus D$ with the boundary analytically parametrized by $\xi^{-1}|_{S^1} : S^1 \rightarrow \partial(\Sigma \setminus D)$.

We now sew $\overline{D^\epsilon}$ and $\Sigma \setminus D$ along their boundaries by identifying $x \in \partial(\Sigma \setminus D)$ with $x' \in \partial\overline{D^\epsilon}$ if and only if $x' = (v^\epsilon \circ \xi)(x)$. Let

$$\Sigma^\epsilon = (\Sigma \setminus D) \sqcup \overline{D^\epsilon} / \text{boundary identification}$$

and we say this Riemann surface is obtained from Σ by *Schiffer variation* of complex structure on D . Let

$$\iota^\epsilon : \Sigma \setminus D \longrightarrow \Sigma^\epsilon \quad \text{and} \quad \iota_D^\epsilon : D^\epsilon \longrightarrow \Sigma^\epsilon$$

be the holomorphic inclusion maps. With a slight abuse of notation we could use the identity map in place of ι^ϵ , however the extra notation will make the following exposition clearer.

Define $w^\epsilon : \mathbb{D} \rightarrow \overline{D^\epsilon}$ by $w^\epsilon(z) = z + \epsilon\bar{z}$. Note that w^ϵ is a homeomorphism, and on the boundary $v^\epsilon = w^\epsilon$. Define the quasiconformal homeomorphism $\nu^\epsilon : \Sigma \rightarrow \Sigma^\epsilon$ by

$$\nu^\epsilon(x) = \begin{cases} \iota^\epsilon(x), & x \in \Sigma \setminus D \\ (\iota_D^\epsilon \circ w \circ \xi)(x), & x \in D. \end{cases}$$

So we now have a point $[\Sigma, \nu^\epsilon, \Sigma^\epsilon] \in T(\Sigma)$ obtained by Schiffer variation of the base point $[\Sigma, \text{id}, \Sigma]$.

To get coordinates on $T(\Sigma)$ we perform Schiffer variation on d disks where $d = 3g - 3 + n$ is the complex dimension of $T(\Sigma)$. Let (D_1, \dots, D_d) be d disjoint parametric disks on Σ , where $D_i = (\xi_i)^{-1}(\mathbb{D})$ for suitably chosen local coordinates ξ_i . Let $D = D_1 \cup \dots \cup D_d$ and let $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \mathbb{C}^d$. Schiffer variation can be performed on the d disks to get a new surface which we again denote by Σ^ϵ . The map ν^ϵ becomes

$$(4.1) \quad \nu^\epsilon(x) = \begin{cases} \iota^\epsilon(x), & x \in \Sigma \setminus D \\ (\iota_D^\epsilon \circ w^{\epsilon_i} \circ \xi_i)(x), & x \in D_i, i = 1, \dots, d. \end{cases}$$

The following theorem is the main result on Schiffer variation [10, 21]. Let $\Omega \subset \mathbb{C}^d$ be an open neighborhood of 0 such that Schiffer variation is defined for $\epsilon \in \Omega$. Define

$$(4.2) \quad \begin{aligned} \mathcal{S} : \Omega &\longrightarrow T(\Sigma) \\ \epsilon &\longmapsto [\Sigma, \nu^\epsilon, \Sigma^\epsilon]. \end{aligned}$$

Theorem 4.12. *Given any d disjoint parametric disks on Σ , it is possible to choose the local parameters ξ_i such that on some open neighborhood Ω of $0 \in \mathbb{C}^n$, $\mathcal{S} : \Omega \rightarrow \mathcal{S}(\Omega)$ is a biholomorphism. That is, the parameters $(\epsilon_1, \dots, \epsilon_d)$ provide local holomorphic coordinates for Teichmüller space in a neighborhood of $[\Sigma, \text{id}, \Sigma]$*

It is important to note that we are free to choose the domains D_i on which the Schiffer variation is performed.

By a standard change of base point argument we can use Schiffer variation to produce a neighborhood of any point $[\Sigma, f, \Sigma_1] \in T(\Sigma)$. Performing Schiffer variation on Σ_1 gives a neighborhood $\mathcal{S}(\Omega)$ of $[\Sigma_1, \text{id}, \Sigma_1] \in T(\Sigma_1)$. Consider the change of base point biholomorphism (see [21, Sections 2.3.1 and 3.2.5]) $f^* : T(\Sigma_1) \rightarrow T(\Sigma)$ given by $f^*([\Sigma_1, g, \Sigma_2]) = [\Sigma, g \circ f, \Sigma_2]$. Then $f^* \circ \mathcal{S}$ is a biholomorphism onto its image $f^*(\mathcal{S}(\Omega)) = \{[\Sigma, \nu^\epsilon \circ f, \Sigma_1^\epsilon]\}$ which is a neighborhood of $[\Sigma, f, \Sigma_1] \in T(\Sigma)$.

Thus, denoting $f^* \circ \mathcal{S}$ itself by \mathcal{S} , the Schiffer variation

$$(4.3) \quad \begin{aligned} \mathcal{S} : \Omega &\longrightarrow T(\Sigma) \\ \epsilon &\longmapsto [\Sigma, \nu^\epsilon \circ f, \Sigma^\epsilon]. \end{aligned}$$

produces a neighborhood of $[\Sigma, f, \Sigma_1] \in T(\Sigma)$.

4.4. Marked Schiffer family. Fix a point $[\Sigma, f, \Sigma_1] \in T(\Sigma)$. We will show that Schiffer variation on Σ_1 produces a marked holomorphic family of Riemann surfaces with fiber Σ_1^ϵ over the point ϵ and marking $\nu^\epsilon \circ f$. Since this construction does not appear in the literature, we present it here in some detail as it is an essential ingredient in our later proofs. An efficient way to describe the family is to do the sewing for all ϵ simultaneously.

For $i = 1, \dots, d$, let Ω_i be connected open neighborhoods of $0 \in \mathbb{C}$ such that $\Omega = \Omega_1 \times \dots \times \Omega_d$ is an open subset of \mathbb{C}^d for which Schiffer variation is defined and Theorem 4.12 implies that $\mathcal{S} : \Omega \rightarrow \mathcal{S}(\Omega) \subset T(\Sigma)$ is a biholomorphism.

Define, for each $i = 1, \dots, d$,

$$\begin{aligned} w_i : \Omega_i \times \mathbb{D} &\longrightarrow \mathbb{C} \times \mathbb{C} \\ (\epsilon_i, z) &\longmapsto (\epsilon_i, w^{\epsilon_i}(z)), \end{aligned}$$

$$\begin{aligned} v_i : \Omega_i \times A_r^1 &\longrightarrow \mathbb{C} \times \mathbb{C} \\ (\epsilon_i, z) &\longmapsto (\epsilon_i, v^{\epsilon_i}(z)), \end{aligned}$$

and let

$$Y_i = w_i(\Omega_i \times \mathbb{D}).$$

Since w_i is a homeomorphism, Y_i is open and so inherits a complex manifold structure from $\mathbb{C} \times \mathbb{C}$. Note that for fixed ϵ_i , $\{z \mid (\epsilon_i, z) \in Y_i\} = D^{\epsilon_i}$.

With $r < 1$ as in the construction of Schiffer variation, let $D_i^r = \xi_i^{-1}(B(0, r))$ and $D^r = D_1^r \cup \dots \cup D_d^r$. Let

$$X = \Omega \times (\Sigma_1 \setminus \overline{D^r})$$

and endow it with the product complex manifold structure. Define the map

$$\begin{aligned} \rho_i : \Omega \times (D_i \setminus \overline{D_i^r}) &\longrightarrow v(\Omega \times A_r^1) \\ (\epsilon_i, x) &\longmapsto (\epsilon, v^{\epsilon_i}(\xi_i(x))). \end{aligned}$$

From the definition of v^{ϵ_i} it follows directly that ρ_i is a biholomorphism from an open subset of X to an open subset of Y_i .

Using the standard gluing procedure for complex manifolds (see for example [7, page 170]) we can make the following definition.

Definition 4.13. Let $S(\Omega, D)$ be the complex manifold obtained by gluing X to Y_1, \dots, Y_d using the biholomorphisms ρ_1, \dots, ρ_d .

The inclusions $\iota_X : X \hookrightarrow S(\Omega, D)$ and $\iota_{Y_i} : Y_i \hookrightarrow S(\Omega, D)$ are holomorphic. Moreover, since r just determines the size of the overlap, $S(\Omega, D)$ is independent of r .

Equivalently, we can think of gluing $\Omega \times (\Sigma_1 \setminus D)$ and $w(\Omega \times \mathbb{D})$ using the ρ_i restricted to $\Omega \times \partial D_i$ to identify the boundary components. For each fixed ϵ this gluing is precisely that used to define Σ_1^ϵ . So we see that

$$S(\Omega, D) = \{(\epsilon, x) : \epsilon \in \Omega, x \in \Sigma_1^\epsilon\}.$$

Define the projection map

$$\begin{aligned} \pi_S : S(\Omega, D) &\rightarrow \Omega \\ (\epsilon, x) &\mapsto \epsilon \end{aligned}$$

and the trivialization

$$(4.4) \quad \begin{aligned} \theta : \Omega \times \Sigma &\rightarrow S(\Omega, D) \\ (\epsilon, x) &\mapsto (\epsilon, (\nu^\epsilon \circ f)(x)). \end{aligned}$$

It is immediate that π_S is onto, holomorphic and defines a topologically trivial bundle.

Definition 4.14. We call $\pi_S : S(\Omega, D) \rightarrow \Omega$ with trivialization θ a marked Schiffer family.

We will have use for explicit charts on $S(\Omega, D)$, but only on the part that is disjoint from the Schiffer variation. Let (U, ζ) be a chart on $\Sigma_1 \setminus \overline{D}$. Recall that $\iota^\epsilon : \Sigma_1 \setminus \overline{D} \rightarrow \Sigma_1^\epsilon$ is the holomorphic inclusion map. Let

$$(4.5) \quad \tilde{U} = \{(\epsilon, x) \mid \epsilon \in \Omega, x \in \iota^\epsilon(U)\} \subset S(\Omega, D)$$

and define

$$(4.6) \quad \begin{aligned} \tilde{\zeta} : \tilde{U} &\longrightarrow \mathbb{C} \times \mathbb{C} \\ (\epsilon, x) &\longmapsto (\epsilon, (\zeta \circ (\iota^\epsilon)^{-1})(x)). \end{aligned}$$

Then $(\tilde{\zeta}, \tilde{U})$ is a holomorphic chart on $S(\Omega, D)$.

Note that with a slight of abuse of notation we could simply write $\tilde{U} = \Omega \times U$ and define $\tilde{\zeta}$ by $(\epsilon, x) \mapsto (\epsilon, \zeta(x))$, but we will refrain from doing so.

Theorem 4.15. *A marked Schiffer family is a marked holomorphic family of Riemann surfaces.*

Proof. We must check the conditions in Definitions 4.5 and 4.7.

Because ν^ϵ is a quasiconformal homeomorphism, $\theta(\epsilon, z)$ is a homeomorphism, and for fixed ϵ , $\theta(\epsilon, z)$ is quasiconformal. Next, we show that for fixed x , $\theta(\epsilon, x)$ is holomorphic in ϵ .

- (1) If $x \in \Sigma \setminus f^{-1}(D_i)$ then $\theta(\epsilon, x) \in \iota_X(X)$. Let ζ and ζ' be a local coordinates in neighborhoods of x and $f(x)$ respectively, and let $z = \zeta(x)$. Use these to form the product charts on $\Omega \times \Sigma$ and X . From the definition of ν^ϵ (see (4.1)) it follows directly that in terms of local coordinates $\theta(\epsilon, x)$ is the map $(\epsilon, z) \mapsto (\epsilon, (\zeta' \circ f \circ \zeta^{-1})(z))$. Since the second entry is independent of ϵ the map is clearly holomorphic in ϵ .
- (2) If $x \in f^{-1}(D_i)$ then $\theta(\epsilon, x) \in \iota_{Y_i}(Y_i)$. Let η be a coordinate map on $f^{-1}(D_i)$ and let $z = \eta(x)$. Use $(\epsilon, t) \mapsto (\epsilon, \zeta(t))$ as the product chart on $\Omega \times f^{-1}(D_i)$. Let $y = \xi_i \circ f \circ \eta^{-1}(z)$ which is independent of ϵ . Then in terms of local coordinates, θ becomes $(\epsilon, z) \mapsto (\epsilon, w^{\epsilon_i}(y))$. Since $w^{\epsilon_i}(y) = y + \epsilon_i \bar{y}$, it is certainly holomorphic in ϵ for fixed y .

Conditions (1) and (2) of Definition 4.7 are thus satisfied. It remains to prove condition (2) of Definition 4.5.

Because $\theta(\epsilon, x)$ is holomorphic in ϵ , $S(\Omega, D)$ has a holomorphic section through every point. This implies that $\pi_S : S(\Omega, D) \rightarrow \Omega$ is a holomorphic split submersion (see for example [21, section 1.6.2], and also [17, Section 6.2] for an alternate definition of marked families).

So $\theta(\epsilon, z)$ is a strong trivialization and hence $S(\Omega, D)$ is a marked family of Riemann surfaces. \square

We will need the following lemma regarding maps between marked Schiffer families. We consider two Schiffer families, whose corresponding neighborhoods in Teichmüller space intersect on an open set and the morphism between these families.

For $i = 1, 2$, let $\pi_i : S_i(\Omega_i, D_i) \rightarrow \Omega_i$ be marked Schiffer families based at $[\Sigma, f_i, \Sigma_i]$. Let $\mathcal{S}_i : \Omega_i \rightarrow T(\Sigma)$ be the corresponding variation maps defined by (4.3), and assume that $\mathcal{S}_1(\Omega_1) \cap \mathcal{S}_2(\Omega_2)$ is non-empty. Let N be any connected component of $\mathcal{S}_1(\Omega_1) \cap \mathcal{S}_2(\Omega_2)$, and let $\Omega'_i = \mathcal{S}_i^{-1}(N)$.

Consider the marked Schiffer families $S_i(\Omega'_i, D_i) = \pi_i^{-1}(\Omega'_i)$ with trivializations $\theta_i : \Omega'_i \times \Sigma \rightarrow S_i(\Omega'_i, D_i)$ defined by $\theta_i(\epsilon, x) = (\epsilon, (\nu_i^\epsilon \circ f_i)(x))$. For ease of notation we write $S'_i = S_i(\Omega'_i, D_i)$.

Recall that throughout we are assuming that Σ is of type (g, n) with $2g - 2 + n > 0$.

Lemma 4.16. *There is a unique invertible morphism of marked families (α, β) from $\pi_1 : S'_1 \rightarrow \Omega'_1$ to $\pi_2 : S'_2 \rightarrow \Omega'_2$. In particular, the following hold:*

- (1) *There is a unique map $\alpha : \Omega'_1 \rightarrow \Omega'_2$ such that $[\Sigma, \nu_1^\epsilon \circ f_1, \Sigma_1^\epsilon] = [\Sigma, \nu_2^{\alpha(\epsilon)} \circ f_2, \Sigma_2^{\alpha(\epsilon)}]$, and α is a biholomorphism.*

- (2) For each $\epsilon \in \Omega'_1$, there is a unique biholomorphism $\sigma_\epsilon : \Sigma_1^\epsilon \rightarrow \Sigma_2^{\alpha(\epsilon)}$ realizing the equivalence in (1).
- (3) The function $\beta(\epsilon, z) = (\alpha(\epsilon), \sigma_\epsilon(z))$ is a biholomorphism on $\pi_1^{-1}(\Omega'_1) \subset S_1(\Omega_1, D_1)$.

Proof. By Theorem 4.11 there are unique mappings of marked families (α_i, β_i) from $\pi_i : S'_i \rightarrow \Omega'_i$ to $\pi_T : \mathcal{T}(\Sigma) \rightarrow T(\Sigma)$. By Theorem 4.12 and the fact that $\alpha_i = \mathcal{S}_i$ from equation (4.3) we see that α_i is injective. Since β_i is injective fibrewise and $\alpha_i \circ \pi_i = \pi_T \circ \beta_i$ it follows that β_i is injective. So α_i and β_i are biholomorphisms onto their images since they are holomorphic and injective functions on finite-dimensional complex spaces.

Let $\alpha = \alpha_2^{-1} \circ \alpha_1$ and $\beta = \beta_2^{-1} \circ \beta_1$; these are biholomorphisms from $\Omega'_1 \rightarrow \Omega'_2$ and $S'_1 \rightarrow S'_2$ respectively. Then (α, β) is the unique map of marked families from $\pi_1 : S'_1 \rightarrow \Omega'_1$ to $\pi_2 : S'_2 \rightarrow \Omega'_2$, and has inverse $(\alpha^{-1}, \beta^{-1})$.

The proof of (1) is completed by noting that the equation

$$[\Sigma, \nu_1^\epsilon \circ f_1, \Sigma_1^\epsilon] = [\Sigma, \nu_2^{\alpha(\epsilon)} \circ f_2, \Sigma_2^{\alpha(\epsilon)}]$$

is precisely $\alpha_1(\epsilon) = \alpha_2(\alpha(\epsilon))$, which is true by the definition of α .

Because β restricted to the fibres is a biholomorphism and $\alpha_1 \circ \pi_1 = \pi_2 \circ \beta$ we can write (as in Remark 4.10) β in the form

$$\beta(\epsilon, x) = (\alpha(\epsilon), \sigma_\epsilon(x))$$

where $\sigma_\epsilon : \Sigma_1^\epsilon \rightarrow \Sigma_2^{\alpha(\epsilon)}$ is a biholomorphism.

Since $2g - 2 + n > 0$, the uniqueness in (2) follows directly from Theorem 4.3.

We have already proved that $\beta : S'_1 \rightarrow S'_2$ is a biholomorphism and so (3) is proved. \square

Remark 4.17. Part (3) of the above lemma is the reason for introducing the theory of marked families. Without this theory, it is impossible to prove (or even formulate the notion of) holomorphicity in ϵ of the map σ_ϵ realizing the Teichmüller equivalence. The holomorphicity in ϵ is necessary for the proof that the transition functions on the rigged Teichmüller space are biholomorphisms (Theorem 4.27 below).

4.5. Topology and atlas for the rigged Teichmüller space. We will now give the rigged Teichmüller space a Hilbert manifold structure.

We begin by defining a base for the topology. Let Σ be a punctured Riemann surface of type (g, n) . We fix a point $[\Sigma, f, \Sigma_1] \in T(\Sigma)$. Let (ζ, E) be an n -chart on Σ_1 , let $U \subset \mathcal{O}_0^{\text{qc}} \times \cdots \times \mathcal{O}_0^{\text{qc}}$ be compatible with (ζ, E) , and let $V = V_{\zeta, E, U}$ (defined in equation (3.16)).

Definition 4.18. We say that a marked Schiffer family $S(\Omega, D)$ is compatible with an n -chart (ζ, E) if the closure of each disc D_i is disjoint from the closure of E_j for all i and j .

For any punctured Riemann surface Σ' denote by $\mathcal{V}(\Sigma')$ the basis of $\mathcal{O}_0^{\text{qc}}(\Sigma')$ as in Definition 3.12.

Lemma 4.19. Let $S(\Omega, D)$ be a marked Schiffer family based at $[\Sigma, f, \Sigma_1]$ and let $V \in \mathcal{V}(\Sigma_1)$. If $S(\Omega, D)$ is compatible with V then $\nu^\epsilon(V) = \{\nu^\epsilon \circ \phi : \phi \in V\}$ is an element of $\mathcal{V}(\Sigma_1^\epsilon)$.

Proof. Writing V in terms of its corresponding n -chart (ζ, E) and $W \subset \mathcal{O}_0^{\text{qc}} \times \cdots \times \mathcal{O}_0^{\text{qc}}$, this is an immediate consequence of the fact that ν^ϵ is holomorphic on the sets E_i . \square

Define the set

$$F(V, S, \Delta) = \{[\Sigma, \nu^\epsilon \circ f, \Sigma_1^\epsilon, \phi] : \epsilon \in \Delta, \phi \in \nu^\epsilon(V)\}$$

where $V \in \mathcal{V}$, $S = S(\Omega, D)$ is a Schiffer variation compatible with V , and Δ is a connected open subset of Ω . The base \mathcal{F} consists of such sets.

Definition 4.20. The base for the topology of $\tilde{T}_0(\Sigma)$ is

$$\mathcal{F} = \{F(V, S, \Delta) : S(\Omega, D) \text{ compatible with } V, \Delta \subseteq \Omega \text{ open and connected}\}.$$

It is an immediate consequence of the definition that the restriction of any $F \in \mathcal{F}$ to a fibre is open in the following sense.

Lemma 4.21. *Let Σ and \mathcal{F} be as above. For any $F \in \mathcal{F}$ and representative (Σ, f_1, Σ_1) of any point $[\Sigma, f_1, \Sigma_1] \in T(\Sigma)$*

$$\{\phi \in \mathcal{O}_0^{\text{qc}}(\Sigma_1) : [\Sigma, f_1, \Sigma_1, \phi] \in F\}$$

is an open subset of $\mathcal{O}_0^{\text{qc}}(\Sigma_1)$.

Proof. This follows immediately from Lemma 4.19. \square

It is necessary to show that \mathcal{F} is indeed a base. This will be accomplished in several steps, together with the proof that the overlap maps of the charts are biholomorphisms. The charts are given in the following definition.

Definition 4.22. For each open set $F(V, S, \Delta) \subset \tilde{T}_0(\Sigma)$ we define the chart

$$(4.7) \quad \begin{aligned} G : \Delta \times U &\longrightarrow F(V, S, \Delta) \\ (\epsilon, \psi) &\longmapsto [\Sigma, \nu^\epsilon \circ f_1, \Sigma_1^\epsilon, \nu^\epsilon \circ \zeta^{-1} \circ \psi]. \end{aligned}$$

where $U \subset (\mathcal{O}_0^{\text{qc}})^n$ is related to V as in Definition 3.12 and $S = S(\Omega, D)$ is compatible with V .

Lemma 4.23. *The map G is a bijection.*

Proof. If $G(\epsilon_1, \psi_1) = G(\epsilon_2, \psi_2)$ then $\epsilon_1 = \epsilon_2$ by Theorem 4.12. Because $2g - 2 + n > 0$, Corollary 4.4 implies $\psi_1 = \psi_2$. This proves injectivity. Surjectivity of G follows from the definition of $F(V, S, \Delta)$. \square

It was shown in [25], that if in the above map $\mathcal{O}_0^{\text{qc}}$ and $\mathcal{O}_0^{\text{qc}}(\Sigma)$ are replaced by \mathcal{O}^{qc} and $\mathcal{O}^{\text{qc}}(\Sigma)$, and the corresponding changes are made to the sets U_i and V_i , then these coordinates can be used to form an atlas on $\tilde{T}^P(\Sigma)$. We need to show the same result in the refined setting.

Remark 4.24. Between here and the end of the proof of Lemma 4.25, we will suppress the subscripts on n -charts (ζ_i, E_i) and elements of $\mathcal{O}_0^{\text{qc}}(\Sigma_1)$ to avoid clutter. The subscripts which remain will distinguish n -charts on different Riemann surfaces.

When clarification is necessary we will use the notation, for example $(\zeta_{i,j}, E_{i,j})$, where the first index labels the Riemann surface and the second labels the puncture.

We proceed as follows. We first prove two lemmas, whose purpose is to show that in a neighborhood of any point, the transition functions are defined and holomorphic on some open set. Once this is established, we show that \mathcal{F} is a base, the topology is Hausdorff and separable, and the charts form a holomorphic atlas.

Some notation is necessary regarding the transition functions. Fix two points $[\Sigma, f_1, \Sigma_1]$ and $[\Sigma, f_2, \Sigma_2]$ in $T(\Sigma)$. Let G_1 and G_2 be two corresponding parametrizations as in (4.7) above, defined on $\Delta_1 \times U_1$ and $\Delta_2 \times U_2$ respectively and using the two Schiffer families $S_1(\Delta_1, D_1)$ and $S_2(\Delta_2, D_2)$. We assume that the intersection $G_1(\Delta_1 \times U_1) \cap G_2(\Delta_2 \times U_2)$ is non-empty.

From the definitions of $\widetilde{T}_0(\Sigma)$ and \mathcal{S} it follows that $\mathcal{S}(\Delta_1) \cap \mathcal{S}(\Delta_2)$ is also non-empty. We follow the notation and setup of Lemma 4.16 and the paragraph immediately preceding it, with $\Delta'_i = \mathcal{S}_i^{-1}(N)$ replacing Ω'_i , where N is any connected component of $\mathcal{S}(\Delta_1) \cap \mathcal{S}(\Delta_2)$.

Recall that in $\widetilde{T}_0^P(\Sigma)$, $[\Sigma, g_1, \Sigma_1, \phi_1] = [\Sigma, g_2, \Sigma_2, \phi_2]$ if and only if $[\Sigma, g_1, \Sigma_1] = [\Sigma, g_2, \Sigma_2]$ via the biholomorphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ and $\sigma \circ \phi_1 = \phi_2$. Lemma 4.16 now implies that $G_1(\epsilon, \psi) = G_2(\epsilon', \psi')$ if and only if $\epsilon' = \alpha(\epsilon)$ and

$$\nu_2^{\alpha(\epsilon)} \circ \zeta_2^{-1} \circ \psi' = \sigma_\epsilon \circ \nu_1^\epsilon \circ \zeta_1^{-1} \circ \psi.$$

Let

$$(4.8) \quad \mathcal{H}(\epsilon, z) = \mathcal{H}_\epsilon(z) = \left(\zeta_2 \circ (\nu_2^{\alpha(\epsilon)})^{-1} \circ \sigma_\epsilon \circ \nu_1^\epsilon \circ \zeta_1^{-1} \right)(z)$$

which is a function of two complex variables. We also define

$$\mathcal{G}(\epsilon, z) = (\alpha(\epsilon), \mathcal{H}(\epsilon, z)).$$

Note that this is shorthand for a collection of maps $\mathcal{H}^j(\epsilon, z)$ and $\mathcal{G}^j(\epsilon, z)$, $j = 1, \dots, n$, where j indexes the punctures (cf. Remark 4.24). Define further

$$(4.9) \quad \begin{aligned} H : \Omega'_1 \times (\mathcal{O}_0^{\text{qc}})^n &\longrightarrow (\mathcal{O}_0^{\text{qc}})^n \\ (\epsilon, \psi) &\longmapsto \mathcal{H}_\epsilon \circ \psi. \end{aligned}$$

The overlap maps can then be written

$$(4.10) \quad (G_2^{-1} \circ G_1)(\epsilon, \psi) = (\alpha(\epsilon), \mathcal{H}_\epsilon \circ \psi) = (\alpha(\epsilon), H(\epsilon, \psi)).$$

Lemma 4.25. *Let $[\Sigma, f_1, \Sigma_1]$ and $[\Sigma, f_2, \Sigma_2] \in \widetilde{T}_0^P(\Sigma)$ for a punctured Riemann surface Σ . For $i = 1, 2$ let \mathcal{V}_i be the base for the topology on $\mathcal{O}_0^{\text{qc}}(\Sigma_i)$ as in Definition 3.12. Again for $i = 1, 2$ let (ζ_i, E_i) be n -charts on Σ_i , let $V_i \in \mathcal{V}_i$ be compatible with the n -charts (ζ_i, E_i) , and let $S_i(\Omega_i, D_i)$ be Schiffer variations compatible with V_i . Finally, for open connected sets $\Delta_i \subseteq \Omega_i$ consider the sets $F(V_i, S_1, \Delta_i)$ which we assume have non-empty intersection.*

Choose any $e_1 \in \Delta_1$ and $\phi_1 \in V_1$ such that $[\Sigma, \nu_1^{e_1} \circ f_1, \Sigma_1^{e_1}, \nu_1^{e_1} \circ \phi_1] \in F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$. Then, there exists a $\Delta \subset \mathcal{S}_1^{-1}(\mathcal{S}_1(\Delta_1) \cap \mathcal{S}_2(\Delta_2))$ containing e_1 , and an open set $E'_1 \subseteq \zeta_1(E_1)$ containing $\overline{\zeta_1 \circ \phi_1(\mathbb{D})}$, such that \mathcal{H} is holomorphic in ϵ and z on $\Delta \times E'_1$ and $\mathcal{G}(\epsilon, z) = (\alpha(\epsilon), \mathcal{H}(\epsilon, z))$ is a biholomorphism onto $\mathcal{G}(\Delta \times E'_1)$.

Proof. Let N be the connected component of $\mathcal{S}_1(\Delta_1) \cap \mathcal{S}_2(\Delta_2)$ that contains $\mathcal{S}_1(e_1)$. For $i = 1, 2$, let $\Delta'_i = \mathcal{S}_i^{-1}(N)$,

$$E_i^{\epsilon_i} = \nu_i^{\epsilon_i}(E_i)$$

and

$$A_i^{\epsilon_i} = (\nu_i^{\epsilon_i} \circ \phi_1)(\mathbb{D}).$$

Note that $\overline{A_i^{\epsilon_i}} \subset E_i^{\epsilon_i}$. By construction Δ'_1 contains e_1 .

Let

$$\widetilde{E}_i = \{(\epsilon_i, z) : \epsilon_i \in \Delta'_i, z \in E_i^{\epsilon_i}\}$$

and

$$\widetilde{A}_i = \{(\epsilon_i, z) : \epsilon_i \in \Delta'_i, z \in A_i^{\epsilon_i}\}.$$

Both of these sets are open by definition of $S(\Omega_i, D_i)$.

Now by Lemma 4.16 there is a biholomorphism $\beta : S(\Delta'_1, D_1) \rightarrow S(\Delta'_2, D_2)$ and moreover, $\beta(\tilde{A}_1) = \tilde{A}_2$. The last assertion follows from the definition of equivalence in the rigged Teichmüller space.

Let

$$\tilde{C} = \beta^{-1}(\tilde{E}_2) \cap \tilde{E}_1$$

and note that $\tilde{A}_1 \subset \tilde{C}$.

Since \tilde{C} is open, so is

$$J = \tilde{\zeta}_1(\tilde{C}) \subset \Delta'_1 \times \zeta_1(E_1),$$

where $\tilde{\zeta}_1$ is defined in (4.6). Let $J^\epsilon = \{z : (\epsilon, z) \in J\}$. Then

$$\overline{\psi_1(\mathbb{D})} \subset J^\epsilon \subset \zeta_1(E_1)$$

for all ϵ , where $\psi_1 = \zeta_1 \circ \phi_1$. By the definition of \tilde{C} , \mathcal{H} is defined on J^ϵ .

We claim that there are connected open sets Δ and E' such that the closure of $\Delta \times E'$ is contained in J , $e_1 \in \Delta$ and $\overline{\psi_1(\mathbb{D})} \subset E'$. Since J is open and $\{e_1\} \times \overline{\psi_1(\mathbb{D})}$ is compact the existence of such sets Δ and E' follow from a standard topological argument.

Since \mathcal{H} , and therefore \mathcal{G} are defined on J they are defined on $\Delta \times E'$. We will prove that \mathcal{G} is biholomorphic by showing that it is equal to β expressed in terms of local coordinates. Using the coordinates defined in (4.6), noting that on E' , $\nu^\epsilon = \iota^\epsilon$, and applying Lemma 4.16, we have for $(\epsilon, z) \in \Delta \times E'$ that

$$\begin{aligned} (\tilde{\zeta}_2 \circ \beta \circ \tilde{\zeta}_1^{-1})(\epsilon, z) &= \left(\alpha(\epsilon), (\zeta_2 \circ (\nu_2^{\alpha(\epsilon)})^{-1} \circ \sigma_\epsilon \circ \nu_1^\epsilon \circ \zeta_1^{-1})(z) \right) \\ &= (\alpha(\epsilon), \mathcal{H}(\epsilon, z)) \\ &= \mathcal{G}(\epsilon, z). \end{aligned}$$

Since β is a biholomorphism we see that on the domain $\Delta \times E'$, \mathcal{G} is a biholomorphism and \mathcal{H} is holomorphic. \square

Theorem 4.26. *With notation as in Lemma 4.25, assume that $p = [\Sigma, \nu_1^{\epsilon_1} \circ f_1, \Sigma_1^{\epsilon_1}, \nu_1^{\epsilon_1} \circ \phi_1]$ is an arbitrary point in $F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$. There exists a $V'_1 \in \mathcal{V}_1$ and a Δ'_1 such that*

- (1) $p \in F(V'_1, S_1, \Delta'_1) \subseteq F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$
- (2) For all $\psi \in U'_1$ (where U'_1 is associated to V'_1 as in Definition 3.12), $\overline{\psi(\mathbb{D})}$ is contained in an open set E' satisfying the consequences of Lemma 4.25
- (3) $G_2^{-1} \circ G_1$ is holomorphic on $\Delta'_1 \times U'_1$.

Proof. By Lemma 4.25, there is an open set $\Delta'_1 \times E'_1$ such that $\overline{\zeta_1 \circ \phi_1(\mathbb{D})} \subset E'_1$, $e_1 \in \Delta'_1$, \mathcal{H} is holomorphic on $\Delta'_1 \times E'_1$ and \mathcal{G} is biholomorphic on $\Delta'_1 \times E'_1$. This immediately implies that there is an open set $\Delta'_2 \times E'_2 \subset \mathcal{G}(\Delta'_1 \times E'_1)$ such that $\alpha(e_1) \in \Delta'_2$ and for $\psi_2 = H(e_1, \zeta_1 \circ \phi_1)$, $\overline{\psi_2(\mathbb{D})} \subseteq E'_2$. Now let $W_2 = \{\psi \in \mathcal{O}_0^{\text{qc}} : \overline{\psi(\mathbb{D})} \subseteq E'_2\}$. By Theorem 3.4 and Remark 3.19, $W_2 \cap U_2$ is open in $\mathcal{O}_0^{\text{qc}}$. Note that $H(e_1, \zeta_1 \circ \phi_1) \in W_2 \cap U_2$.

Choose a compact set $K \subset E'_1$ which contains $\overline{\zeta_1 \circ \phi_1(\mathbb{D})}$ in its interior K_{int} . If we let $W_1 = \{\psi \in \mathcal{O}_0^{\text{qc}} : \overline{\psi(\mathbb{D})} \subseteq K_{\text{int}}\}$, then W_1 is open by Theorem 3.4. We claim that H is holomorphic on $\Delta'_1 \times W_1$. By Hartogs' theorem (see [19] for a version in a suitably general setting), it is enough to check holomorphicity separately in ϵ and ψ . By Lemma 2.6, H is holomorphic in ϵ for fixed ψ . On the other hand, by Theorem 3.9, H is holomorphic in ψ for fixed ϵ by our careful choice of W_1 .

In particular, H is continuous and therefore $H^{-1}(W_2 \cap U_2) \cap (\Delta_1'' \times (W_1 \cap U_1))$ is open and contains $(e_1, \zeta_1 \circ \phi_1)$, hence we may choose an open subset $\Delta_1' \times U_1'$ containing $(e_1, \zeta_1 \circ \phi_1)$. Let V_1' be the element of \mathcal{V}_1 associated to U_1' . Clearly $U_1' \subseteq U_1$, and $H(\Delta_1' \times U_1') \subseteq U_2$ by construction; thus $F(V_1', S_1, \Delta_1') \subseteq F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$ so the first condition is satisfied. By construction, (2) is also satisfied. Since $U_1' \subseteq W_1$, H is holomorphic on $\Delta_1' \times U_1'$ and the fact that α is holomorphic on Δ' yields that $G_2^{-1} \circ G_1$ is holomorphic on $\Delta_1' \times U_1'$. This concludes the proof. \square

Theorem 4.27. *The set \mathcal{F} is a base for a Hausdorff, separable topology on $\tilde{T}_0(\Sigma)$. Furthermore, with the atlas of charts given by (4.7), $\tilde{T}_0(\Sigma)$ is a Hilbert manifold.*

Proof. It follows directly from part (1) of Theorem 4.26 that \mathcal{F} is a base for a topology on $\tilde{T}_0(\Sigma)$. From part (3), we have that the inverses of the maps (4.7) form an atlas with holomorphic transition functions. Thus it remains only to show that this topology is Hausdorff and separable. We first show that it is Hausdorff.

For $i = 1, 2$, let $p_i = [\Sigma, \nu_i^{e_i} \circ f_i, \Sigma_i^{e_i}, \nu_i^{e_i} \circ \phi_i]$ be distinct points, in sets $F(V_i, S_i, \Delta_i)$. If $F(V_i, S_i, \Delta_i)$ are disjoint, we are done. If not, by Lemma 4.16 setting Δ_i' to be the connected component of $\mathcal{S}_i^{-1}(\mathcal{S}_1(\Delta_1) \cap \mathcal{S}_2(\Delta_2))$ containing e_i , there is a biholomorphism $\alpha : \Delta_1' \rightarrow \Delta_2'$ such that $[\Sigma, \nu_2^{\alpha(e_1)} \circ f_2, \Sigma_2^{\alpha(e_1)}] = [\Sigma, \nu_1^{e_1} \circ f_1, \Sigma_1^{e_1}]$ for all $e_1 \in \Delta_1'$.

There are two cases to consider. If $[\Sigma, \nu_2^{\alpha(e_1)} \circ f_2, \Sigma_2^{\alpha(e_1)}] \neq [\Sigma, \nu_2^{e_2} \circ f_2, \Sigma_2^{e_2}]$, then one can find $\Omega_1 \subset \Delta_1$ and $\Omega_2 \subset \Delta_2$ such that $\mathcal{S}_1(\Delta_1)$ and $\mathcal{S}_2(\Delta_2)$ are disjoint and $F(V_i, S_i, \Omega_i)$ still contains $[\Sigma, \nu_1^{e_i} \circ f_1, \Sigma_1^{e_i}, \nu_1^{e_i} \circ \phi_i]$ for $i = 1, 2$. But then $F(V_i, S_i, \Omega_i)$ are disjoint, which takes care of the first case.

If on the other hand $[\Sigma, \nu_2^{\alpha(e_1)} \circ f_2, \Sigma_2^{\alpha(e_1)}] = [\Sigma, \nu_2^{e_2} \circ f_2, \Sigma_2^{e_2}]$, then by Theorem 4.26 there are sets $F(V_1', S_1, \Omega_1)$ and $F(V_2', S_1, \Omega_2)$ in $F(V_1, S_1, \Delta_1) \cap F(V_2, S_2, \Delta_2)$ containing p_1 and p_2 respectively. Thus we may write

$$p_1 = [\Sigma, \nu_1^{e_1} \circ f_1, \Sigma_1^{e_1}, \nu_1^{e_1} \circ \psi_1] \quad \text{and} \quad p_2 = [\Sigma, \nu_1^{e_1} \circ f_1, \Sigma_1^{e_1}, \nu_1^{e_1} \circ \psi_2].$$

For $i = 1, 2$, let U_i' be the subsets of $(\mathcal{O}_0^{\text{qc}})^n$ associated with V_i' as in Definition 3.12. Since $\mathcal{O}_0^{\text{qc}}$ is an open subset of a Hilbert space, it is Hausdorff, so there are open sets W_i in U_i' containing p_i for $i = 1, 2$ and such that $W_1 \cap W_2$ is empty. In that case if V_i'' are the elements of \mathcal{V} associated to W_i , then $V_1'' \cap V_2''$ is empty. This in turn implies that $F(V_1'', S_1, \Omega_1) \cap F(V_2'', S_1, \Omega_2)$ is empty which proves the claim in the second case.

We now prove that $\tilde{T}_0(\Sigma)$ is separable. Since $T(\Sigma)$ is a finite dimensional complex manifold it is, in particular, separable. Choose a countable dense subset \mathfrak{A} of $T(\Sigma)$. For each $p = [\Sigma, f_1, \Sigma_1] \in \mathfrak{A}$, choose a specific representative (Σ, f_1, Σ_1) . The space $\mathcal{O}_0^{\text{qc}}(\Sigma_2)$ is second countable and, in particular, it has a countable dense subset $\mathfrak{B}_p(\Sigma_1)$. Now if (Σ, f_2, Σ_2) is any other representative, there exists a unique biholomorphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ (if σ_1 is another such biholomorphism, since by hypothesis $\sigma_1^{-1} \circ \sigma$ is homotopic to the identity and $2g - 2 + n > 0$, it follows from Theorem 4.3 that $\sigma_1^{-1} \circ \sigma$ is the identity). We set

$$\mathfrak{B}_p(\Sigma_2) = \{(\sigma \circ \phi_1, \dots, \sigma \circ \phi_n) : (\phi_1, \dots, \phi_n) \in \mathfrak{B}_p(\Sigma_1)\}.$$

This is easily seen to be itself a countable dense set in $\mathcal{O}_0^{\text{qc}}(\Sigma_2)$ and it is not hard to see that

$$\Upsilon = \{[\Sigma, f_1, \Sigma_1, \psi_1] : [\Sigma, f_1, \Sigma_1] \in \mathfrak{A}, \psi_1 \in \mathfrak{B}_p(\Sigma_1)\}$$

is well-defined. We will show that it is dense. Note that for any fixed $[\Sigma, f_1, \Sigma_1]$, the set of $[\Sigma, f_1, \Sigma_1, \psi_1] \in \Upsilon$ is entirely determined by any particular representative (Σ, f_1, Σ_1) , and so this is a countable set.

Let $F(V, S, \Delta) \in \mathcal{F}$. Since \mathfrak{A} is dense, there is some $[\Sigma, f_2, \Sigma_2] \in \mathfrak{A} \cap S(\Delta)$. For a specific representative (Σ, f_2, Σ_2) there is a $\psi_2 \in \mathcal{O}_0^{\text{qc}}(\Sigma_2)$ such that $[\Sigma, f_2, \Sigma_2, \psi_2] \in F(V, S, \Delta)$. By Lemma 4.21 the set of points in F over $[\Sigma, f_2, \Sigma_2]$ is open. Thus since $\mathfrak{B}_p(\Sigma_2)$ is dense in $\mathcal{O}_0^{\text{qc}}(\Sigma_2)$ there is a $\psi_3 \in \mathfrak{B}_p(\Sigma_2)$ such that $[\Sigma, f_2, \Sigma_2, \psi_3] \in F$. By definition $[\Sigma, f_2, \Sigma_2, \psi_3] \in \Upsilon$, which completes the proof. \square

Remark 4.28. It can be shown that $\tilde{T}_0(\Sigma)$ is second countable. The proof involves somewhat tedious notational difficulties, so we only give a sketch of the proof. No results in this paper depend on second countability of $\tilde{T}_0(\Sigma)$.

Fix a countable basis \mathfrak{D} for $\mathcal{O}_0^{\text{qc}}$. For any $[\Sigma, f_1, \Sigma_1] \in \mathfrak{A}$, choose a representative (Σ, f_1, Σ_1) , and fix the following objects. Let $\mathfrak{C}(\Sigma_1)$ be a countable collection of n -charts on Σ_1 constructed as in the proof of Theorem 3.13. Let $\mathcal{V}_c(\Sigma_1)$ be the countable dense subset of $\mathcal{V}(\Sigma_1)$ corresponding to \mathfrak{D} and $\mathfrak{C}(\Sigma_1)$ as in the proof of Theorem 3.13. Finally, fix a countable base $\mathfrak{B}(\Sigma_1)$ of open sets in Σ_1 .

Now if (Σ, f_2, Σ_2) is any other representative, there is a unique biholomorphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ as in the proof of Theorem 4.27. Transfer each of the preceding objects to Σ_2 by composition with σ in the appropriate way; for example, $\mathfrak{C}(\Sigma_2)$ is the set of n -charts $(\zeta_1 \circ \sigma^{-1}, \sigma(E_1), \dots, \zeta_n \circ \sigma^{-1}, \sigma(E_n))$ and so on. Finally fix a countable base \mathfrak{D} of \mathbb{C}^n (for example, the set of discs of rational radius centered at rational points).

We now define the subset \mathcal{F}_c of \mathcal{F} to be the set of $F(V, S, \Delta) \in \mathcal{F}$ such that

- (1) the variation $S(\Omega)$ is based at a point $[\Sigma, f_1, \Sigma_1] \in \mathfrak{A}$
- (2) $S(\Omega)$ is compatible with some fixed n -chart in $\mathfrak{C}(\Sigma_1)$
- (3) Ω and Δ are both in $\mathfrak{D} \times \dots \times \mathfrak{D}$
- (4) $V \in \mathcal{V}(\Sigma_1)$.

The set \mathcal{F}_c is countable by construction, and does not depend on the choice of representative. It can be shown with some work that \mathcal{F}_c is a base compatible with \mathcal{F} .

4.6. Compatibility with the non-refined rigged Teichmüller space. In [23] the following rigged Teichmüller space was defined.

Definition 4.29. Let $\tilde{T}(\Sigma)$ be defined by replacing $\mathcal{O}_0^{\text{qc}}(\Sigma_1)$ with $\mathcal{O}^{\text{qc}}(\Sigma_1)$ in Definition 4.2.

It was shown in [24] that $\tilde{T}(\Sigma)$ is a complex Banach manifold with charts as in Definition 4.22 with $U \subset (\mathcal{O}^{\text{qc}})^n$, and \mathcal{O}^{qc} replacing $\mathcal{O}_0^{\text{qc}}$ in all the preceding definitions and constructions. Furthermore, the complex structure on \mathcal{O}^{qc} is given by the embedding χ defined by (2.2). We use the same notation for the charts and constructions on $\tilde{T}(\Sigma)$ as for $\tilde{T}_0(\Sigma)$ without further comment.

The complex structures on $\tilde{T}_0(\Sigma)$ and $\tilde{T}(\Sigma)$ are compatible in the following sense.

Theorem 4.30. *The inclusion map $I_T : \tilde{T}_0(\Sigma) \rightarrow \tilde{T}(\Sigma)$ is holomorphic.*

Proof. Choose any point $[\Sigma, f, \Sigma_*, \phi] \in \tilde{T}_0(\Sigma)$. There is a parametrization $G : \Omega \times U \rightarrow \tilde{T}(\Sigma)$ onto a neighborhood of this point (see Definition 4.22). We choose U small enough that ν^ϵ is holomorphic on $\overline{\phi(\mathbb{D})}$ for all $\phi \in U$.

Let $W = \chi^n(U)$ where $\chi^n : \mathcal{O}^{\text{qc}} \times \dots \times \mathcal{O}^{\text{qc}} \rightarrow \bigoplus^n (A_1^\infty(\mathbb{D}) \oplus \mathbb{C})$ is defined by

$$\chi^n(\phi_1, \dots, \phi_n) = (\chi(\phi_1), \dots, \chi(\phi_n)).$$

Define $F : \Omega \times W \rightarrow \tilde{T}(\Sigma)$ by

$$F = G \circ (\text{id}, (\chi^n)^{-1})$$

where id is the identity map on Ω . These are coordinates on $\tilde{T}(\Sigma)$.

Let $W_0 = W \cap \mathcal{O}_0^{\text{qc}} = \iota^{-1}(W)$ (recall that ι is the inclusion map of $\mathcal{O}_0^{\text{qc}}$ in \mathcal{O}^{qc}). The set W_0 is open by Theorem 2.3. We further have that $F(\Omega \times W_0) = \tilde{T}_0 \cap W$. To see this note that $F(\Omega \times W_0) = G(\Omega \times (\chi^n)^{-1}(W_0))$. By definition $\nu^\epsilon \circ \phi \in \mathcal{O}_0^{\text{qc}}(\Sigma)$ if and only if for a parameter $\eta : A \rightarrow \mathbb{C}$ defined on an open neighborhood A of $\nu^\epsilon(\phi(\mathbb{D}))$ it holds that $\eta \circ \nu^\epsilon \circ \phi \in \mathcal{O}_0^{\text{qc}}$. This holds if and only if $\phi \in \mathcal{O}_0^{\text{qc}}$ since ν^ϵ is holomorphic on a neighborhood of $\overline{\phi(\mathbb{D})}$.

It follows from Theorem 3.4 that $F^{-1} \circ I_T \circ F$ is holomorphic. Since F are local coordinates, I_T is holomorphic on the image of F . Since coordinates of the form F cover $\tilde{T}(\Sigma)$, this proves the theorem. \square

Note that this does not imply that $\tilde{T}_0(\Sigma)$ is a complex submanifold of $\tilde{T}(\Sigma)$.

5. A REFINED TEICHMÜLLER SPACE OF BORDERED SURFACES

We are at last in a position to define the refined Teichmüller space of a bordered surface and demonstrate that it has a natural complex Hilbert manifold structure. In Section 5.1 we define the refined Teichmüller space $T_0(\Sigma^B)$ of a bordered surface Σ^B , and define some “modular groups” which act on it. In Section 5.2 we show how to obtain a punctured surface by sewing “caps” onto the bordered surface using the riggings. It is also demonstrated that sewing on caps takes the refined Teichmüller space into the refined rigged Teichmüller space $\tilde{T}_0(\Sigma)$. In Section 5.3 we prove that the refined Teichmüller space of bordered surfaces is a Hilbert manifold. We do this by showing that the refined rigged Teichmüller space $\tilde{T}_0(\Sigma)$ is a quotient of $T_0(\Sigma^B)$ by a properly discontinuous, fixed point free group of local homeomorphisms, and passing the charts on $\tilde{T}_0(\Sigma)$ upwards. Finally, in Section 5.4 we show that the rigged moduli space of Friedan and Shenker is a Hilbert manifold. This follows from the fact that the rigged moduli space is a quotient of $T_0(\Sigma^B)$ by a properly discontinuous fixed-point free group of biholomorphisms.

5.1. Definition of the refined Teichmüller space and modular groups. The reader is referred to Section 2.2 for some of the notation and definitions used below.

We now define the refined Teichmüller space of a bordered Riemann surface which is obtained by replacing the quasiconformal marking maps in the usual Teichmüller space (see Definition 4.1) with refined quasiconformal maps.

Definition 5.1. Fix a bordered Riemann surface Σ^B of type (g, n) . Let

$$T_0(\Sigma^B) = \{(\Sigma^B, f, \Sigma_1^B)\} / \sim$$

where Σ_1^B is a bordered Riemann surface of the same type, $f \in \text{QC}_0(\Sigma^B, \Sigma_1^B)$, and two triples $(\Sigma^B, f_i, \Sigma_i^B)$, $i = 1, 2$ are equivalent if there is a biholomorphism $\sigma : \Sigma_1^B \rightarrow \Sigma_2^B$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel boundary.

The space $T_0(\Sigma^B)$ is called the *refined Teichmüller space* and its elements are denoted by equivalence classes of the form $[\Sigma^B, f_1, \Sigma_1^B]$.

An important ingredient in the construction of the complex Hilbert manifold structure is a kind of modular group (or mapping class group). To distinguish between the different possible boundary condition we use some slightly non-standard notation following [23]; we recall the definitions here.

Let Σ^B be a bordered Riemann surface and $\text{QCI}(\Sigma^B)$ denote the set of quasiconformal maps from Σ^B onto Σ^B which are the identity on the boundary. This is a group which acts on the

marking maps by right composition. Let $\text{QCI}_n(\Sigma^B)$ denote the subset of $\text{QCI}(\Sigma^B)$ which are homotopic to the identity rel boundary (the subscript n stands for “null-homotopic”).

Definition 5.2. Let $\text{PModI}(\Sigma^B) = \text{QCI}(\Sigma^B) / \sim$ where two elements f and g of $\text{QCI}(\Sigma^B)$ are equivalent ($f \sim g$) if and only if $f \circ g^{-1} \in \text{QCI}_n(\Sigma)$.

The “P” stands for “pure”, which means that the mappings preserve the ordering of the boundary components, and “I” stands for “identity”.

There is a natural action of $\text{PModI}(\Sigma^B)$ on $T(\Sigma^B)$ by right composition, namely

$$(5.1) \quad [\rho][\Sigma^B, f, \Sigma_1^B] = [\Sigma^B, f \circ \rho, \Sigma_1^B].$$

This is independent of the choice of representative $\rho \in \text{QCI}(\Sigma^B)$ of $[\rho] \in \text{PModI}(\Sigma^B)$. It is a standard fact that $\text{PModI}(\Sigma^B)$ is finitely generated by Dehn twists. Using these twists we can define two natural subgroups of $\text{PModI}(\Sigma^B)$ (see [23] for details).

Definition 5.3. Let Σ^B be a bordered Riemann surface. Let $\text{DB}(\Sigma^B)$ be the subgroup of $\text{PModI}(\Sigma^B)$ generated by Dehn twists around simple closed curves Σ which are homotopic to a boundary curve. Let $\text{DI}(\Sigma^B)$ be the subgroup of $\text{PModI}(\Sigma^B)$ generated by Dehn twists around simple closed curves in Σ^B which are neither homotopic to a boundary curve nor null-homotopic.

Here “B” stands for “boundary” and “I” stands for “internal”.

The next Lemma implies that we can consider $\text{PModI}(\Sigma^B)$ and $\text{DB}(\Sigma^B)$ as acting on $T_0(\Sigma^B)$.

Lemma 5.4. *Every element of $\text{QCI}(\Sigma^B)$ is in $\text{QC}_0(\Sigma^B, \Sigma^B)$. Thus, the group action of $\text{PModI}(\Sigma^B)$ on $T(\Sigma^B)$ preserves $T_0(\Sigma^B)$.*

Proof. The first statement follows from Definition 2.19, and Definition 2.15 with $H_1 = H_2$. The second statement follows from Proposition 2.20. \square

5.2. Sewing on caps. Given a bordered Riemann surface Σ^B together with quasisymmetric parametrizations of its boundaries by the circle, one can sew on copies of the punctured disc to obtain a punctured Riemann surface Σ . The collection of parametrizations extend to an element of $\mathcal{O}^{\text{qc}}(\Sigma)$. In [23], two of the authors showed that this operation can be used to exhibit a natural correspondence between the rigged Teichmüller space $\tilde{T}(\Sigma)$ and the Teichmüller space $T(\Sigma^B)$, and showed in [25] that this results in a natural fibre structure on $T(\Sigma^B)$. We will be using this fibre structure as the principle framework for constructing the Hilbert manifold structure on $T_0(\Sigma^B)$. It is thus necessary to describe sewing on caps here, in the setting of refined quasisymmetries.

Definition 5.5. Let Σ^B be a bordered Riemann surface with boundary curves C_i , $i = 1, \dots, n$. The *riggings* of Σ^B is the collection $\text{Rig}(\Sigma^B)$ of n -tuples $\psi = (\psi_1, \dots, \psi_n)$ such that $\psi_i \in \text{QS}(S^1, C_i)$. The refined riggings is the collection $\text{Rig}_0(\Sigma^B)$ of n -tuples $\psi = (\psi_1, \dots, \psi_n)$ such that $\psi_i \in \text{QS}_0(S^1, C_i)$.

Let Σ^B be a fixed bordered Riemann surface of type (g, n) say, and $\psi \in \text{Rig}(\Sigma^B)$. Let \mathbb{D}_0 denote the punctured unit disc $\mathbb{D} \setminus \{0\}$. We obtain a new topological space

$$(5.2) \quad \Sigma = \overline{\Sigma^B} \sqcup \overline{\mathbb{D}_0} \sqcup \dots \sqcup \overline{\mathbb{D}_0} / \sim.$$

Here we treat the n copies of \mathbb{D}_0 as distinct and ordered, and two points p and q are equivalent ($p \sim q$) if p is in the boundary of the i th disc, q is in the i th boundary C_i , and

$q = \psi_i(p)$. By [23, Theorems 3.2, 3.3] this topological space has a unique complex structure which is compatible with the complex structures on Σ^B and each copy of \mathbb{D}_0 . We will call the image of a boundary curve in Σ under inclusion (which is also the image of $\partial\mathbb{D}$ under inclusion) a *seam*. We will call the copy of each disc in Σ a *cap*. Finally, we will denote equation (5.2) by

$$\Sigma = \Sigma^B \#_{\psi} \mathbb{D}_0^n$$

to emphasize the underlying element of $\text{Rig}(\Sigma^B)$ used to sew.

For each $i = 1, \dots, n$ the map ψ_i can be extended to a map $\tilde{\psi}_i : \overline{\mathbb{D}_0} \rightarrow \Sigma$ defined by

$$(5.3) \quad \tilde{\psi}_i(z) = \begin{cases} \psi_i(z), & \text{for } z \in \partial\mathbb{D} \\ z, & \text{for } z \in \mathbb{D}. \end{cases}$$

Note that $\tilde{\psi}_i$ is well defined and continuous because the map ψ_i is used to identify $\partial\mathbb{D}$ with C_i . Moreover, $\tilde{\psi}$ is holomorphic on \mathbb{D}_0 . It is important to keep in mind that if the seam in Σ is viewed as $\partial\mathbb{D}$ then in fact $\tilde{\psi}_i$ is also the identity on $\partial\mathbb{D}$.

Remark 5.6. The complex structure on the sewn surface is easily described in terms of conformal welding. Choose a seam C_i and let H be a collared chart (see Definition 2.14) with respect to C_i with domain A say. We have that $H \circ \psi_i$ is in $\text{QS}(S^1)$. Let $F : \mathbb{D} \rightarrow \mathbb{C}$ and $G : \mathbb{D}^* \rightarrow \overline{\mathbb{C}}$ be the unique holomorphic welding maps such that $G^{-1} \circ F = H \circ \psi_i$ when restricted to S^1 , $F(0) = 0$, $G(\infty) = \infty$ and $G'(\infty) = 1$. Note that F and G have quasiconformal extensions to \mathbb{C} and $\overline{\mathbb{C}}$ respectively.

Let ζ_i be the continuous map on $A \cup \tilde{\psi}_i(\mathbb{D})$ defined by

$$(5.4) \quad \zeta_i = \begin{cases} F \circ \tilde{\psi}_i^{-1} & \text{on } \tilde{\psi}_i(\mathbb{D}) \\ G \circ H & \text{on } A. \end{cases}$$

It is easily checked that there is such a continuous extension. Since ζ_i is 0-quasiconformal on $\tilde{\psi}_i(\mathbb{D})$ and A , by removability of quasicircles [18, V.3] ζ_i is 0-quasiconformal (that is, holomorphic and one-to-one), on $A \cup \tilde{\psi}_i(\mathbb{D})$. Thus ζ is a local coordinate on Σ containing the closure of the cap.

The crucial fact about the extension $\tilde{\psi} = (\tilde{\psi}_1, \dots, \tilde{\psi}_n)$ is that it is in $\mathcal{O}_0^{\text{qc}}(\Sigma)$. In fact we have the following proposition.

Proposition 5.7. *Let Σ^B be a bordered Riemann surface, and let $\psi = (\psi_1, \dots, \psi_n)$ be in $\text{QS}(S^1, \Sigma^B)$. Let $\Sigma = \Sigma^B \#_{\psi} \mathbb{D}_0^n$ and $\tilde{\psi} = (\tilde{\psi}_1, \dots, \tilde{\psi}_n)$ be the n -tuple of holomorphic extensions to \mathbb{D}_0 . Then $\psi \in \text{Rig}_0(\Sigma^B)$ if and only if $\tilde{\psi} \in \mathcal{O}_0^{\text{qc}}(\Sigma)$.*

Proof. Let H be a collared chart with respect to the i th boundary curve C_i , and let F , G and ζ_i be as in Remark 5.6. By definition $\psi_i \in \text{QS}_0(S^1, C_i)$ if and only if $H \circ \psi_i \in \text{QS}_0(S^1)$ which holds if and only if the welding map F is in $\mathcal{O}_0^{\text{qc}}$. Since $F = \zeta_i \circ \tilde{\psi}_i$ this proves the claim. \square

The following Proposition is a consequence of Proposition 2.21 and Theorem 2.11.

Proposition 5.8. *Let Σ_1^B and Σ_2^B be bordered Riemann surfaces, and let $\tau \in \text{Rig}_0(\Sigma_1^B)$. Then $f \in \text{QC}_0(\Sigma_1^B, \Sigma_2^B)$ if and only if $f \circ \tau \in \text{Rig}_0(\Sigma_2^B)$.*

We now have enough tools to describe the relation between $T_0(\Sigma^B)$ and $\tilde{T}_0(\Sigma)$.

Definition 5.9. Let Σ^B be a bordered Riemann surface, let $\tau \in \text{Rig}_0(\Sigma)$ be a fixed rigging, and let $\Sigma = \Sigma^B \#_{\tau} \mathbb{D}_0^n$. We define

$$\begin{aligned} \Pi : T(\Sigma^B) &\longrightarrow \tilde{T}(\Sigma) \\ [\Sigma^B, f, \Sigma_1^B] &\longmapsto [\Sigma, \tilde{f}, \Sigma_1, \tilde{f} \circ \tilde{\tau}]. \end{aligned}$$

where $\tilde{\tau}$ is the extension defined by (5.3),

$$(5.5) \quad \tilde{f}(z) = \begin{cases} f(z), & z \in \overline{\Sigma^B} \\ z, & z \in \text{cap}, \end{cases}$$

and $\Sigma_1 = \Sigma_1^B \#_{f \circ \tau} \mathbb{D}_0^n$ is the Riemann surface obtained by sewing caps onto Σ_1^B using $f \circ \tau$.

The map \tilde{f} is quasiconformal, since it is quasiconformal on Σ^B and the cap, and is continuous on the seam [18, V.3].

Remark 5.10. If $\widetilde{f \circ \tau}$ denotes the holomorphic extension of $f \circ \tau$ as in equation (5.3), then $\widetilde{f \circ \tau} = \tilde{f} \circ \tilde{\tau}$.

It was shown in [23] that Π is invariant under the action of DB, and in fact

$$\Pi([\Sigma^B, f, \Sigma_1^B]) = \Pi([\Sigma^B, f_2, \Sigma_2^B]) \iff [\Sigma^B, f_2, \Sigma_2^B] = [\rho][\Sigma^B, f_1, \Sigma_1^B]$$

for some $[\rho] \in \text{DB}$. (The reader is warned that the direction of the riggings in [23] is opposite to the convention used here). Thus $\tilde{T}(\Sigma) = T(\Sigma^B)/\text{DB}$ as sets. Furthermore, the group action by DB is properly discontinuous and fixed point free, and the map Π is holomorphic with local holomorphic inverses. Thus $\tilde{T}(\Sigma)$ inherits a complex structure from $T(\Sigma^B)$.

On the other hand, in the refined setting, instead of having a complex structure on Teichmüller space in the first place, we are trying to construct one. In the next section, we will reverse the argument above and lift the complex Hilbert manifold structure on $\tilde{T}_0(\Sigma)$ to $T_0(\Sigma^B)$. To this end we need the following facts.

Proposition 5.11. *Let $p = [\Sigma^B, f, \Sigma_1^B] \in T(\Sigma^B)$. Then $p \in T_0(\Sigma^B)$ if and only if $\Pi(p) \in \tilde{T}_0(\Sigma)$.*

Proof. Since $\tau \in \text{Rig}_0(\Sigma^B)$, $f \in \text{QC}_0(\Sigma^B, \Sigma_1^B)$ if and only if $f \circ \tau \in \text{Rig}_0(\Sigma_1^B)$ by Proposition 5.8. And this holds if and only if $\widetilde{f \circ \tau} \in \mathcal{O}_0^{\text{qc}}(\Sigma_1)$ by Proposition 5.7. By Remark 5.10, $\tilde{f} \circ \tilde{\tau} \in \mathcal{O}_0^{\text{qc}}(\Sigma_1)$ which proves the claim. \square

We now define the map Π_0 by

$$\Pi_0 = \Pi|_{T_0(\Sigma^B)},$$

and as a result of this proposition we have

$$(5.6) \quad \Pi_0 : T_0(\Sigma^B) \longrightarrow \tilde{T}_0(\Sigma).$$

Proposition 5.12. *The action of DB is fixed point free, and for $[\Sigma^B, f_i, \Sigma_i^B] \in T_0(\Sigma^B)$, $i = 1, 2$, $\Pi_0([\Sigma^B, f_1, \Sigma_1^B]) = \Pi_0([\Sigma^B, f_2, \Sigma_2^B])$ if and only if there is a $[\rho] \in \text{DB}$ such that $[\rho][\Sigma^B, f_1, \Sigma_1^B] = [\Sigma^B, f_2, \Sigma_2^B]$. The map $\Pi : T_0(\Sigma^B) \rightarrow \tilde{T}_0(\Sigma)$ is onto and thus, as sets, $T_0(\Sigma^B)/\text{DB}$ and $\tilde{T}_0(\Sigma)$ are in one-to-one correspondence.*

Proof. These claims are all true in the non-refined setting [23, Lemma 5.1, Theorem 5.6]. Thus by Proposition 5.11 they are true in the refined setting. \square

5.3. Complex Hilbert manifold structure on refined Teichmüller space. Next we describe how to construct the complex structure on $T_0(\Sigma^B)$. Let Σ^B be a bordered Riemann surface, and let $\tau \in \text{Rig}_0(\Sigma^B)$. Let Σ be the Riemann surface obtained by sewing on caps via τ as in the previous section.

We define a base \mathcal{B} for a topology on $T_0(\Sigma^B)$ as follows. Recall that \mathcal{F} is the base for $\tilde{T}_0(\Sigma)$ (Definition 4.20).

Definition 5.13. A set $B \in \mathcal{B}$ if and only if

- (1) $\Pi_0(B) \in \mathcal{F}$
- (2) Π_0 is one-to-one on B .

Theorem 5.14. *The set \mathcal{B} is a base. With the topology corresponding to \mathcal{B} , $\tilde{T}_0(\Sigma)$ has the quotient topology with respect to Π_0 and DB is properly discontinuous.*

Proof. Let $x \in T_0(\Sigma^B)$. We show that there is a $B \in \mathcal{B}$ containing x . There is a neighborhood U of x in $T(\Sigma^B)$ on which Π is one-to-one [23]. Let $U' = \Pi(U)$; this is open in $\tilde{T}(\Sigma)$ [23]. By Theorem 4.30, the set $U' \cap \tilde{T}_0(\Sigma)$ is open in $\tilde{T}_0(\Sigma)$. Thus there is an element $F \subset U' \cap \tilde{T}_0(\Sigma)$ of the base \mathcal{F} which contains $\Pi(x)$. Since $\Pi|_U$ is invertible, we can set $B = (\Pi|_U)^{-1}(F)$, and B is in \mathcal{B} and contains x .

Next, fix $q \in T_0(\Sigma^B)$ and let $B_1, B_2 \in \mathcal{B}$ contain q . We show that the intersection contains an element of \mathcal{B} . Let $U \subset \Pi_0(B_1) \cap \Pi_0(B_2)$ be a set in \mathcal{F} containing $\Pi_0(q)$. Set $B_3 = (\Pi_0|_{B_1})^{-1}(U) \subset B_1 \cap B_2$. We then have that Π_0 is one-to-one on B_3 (since $B_3 \subset B_1$) and $\Pi_0(B_3) = U$. So $B_3 \in \mathcal{B}$. Thus \mathcal{B} is a base.

Now we show that $\tilde{T}_0(\Sigma)$ has the quotient topology with respect to Π_0 . Let U be open in $\tilde{T}_0(\Sigma)$ and let $x \in \Pi_0^{-1}(U)$. There is a $B_x \in \mathcal{B}$ containing x such that Π_0 is one-to-one on B_x , and $\Pi_0(B_x)$ is open and in \mathcal{F} . Since $\Pi_0(B_x) \cap U$ is open and non-empty (it contains $\Pi_0(x)$), there is a $F_x \in \mathcal{F}$ such that $\Pi_0(x) \in F_x$ and $F_x \subset \Pi_0(B_x) \cap U$. By definition $\tilde{B}_x = (\Pi_0|_{B_x})^{-1}(F_x) \in \mathcal{B}$. By construction $x \in \tilde{B}_x$ and \tilde{B}_x is open and contained in U . Since x was arbitrary, $\Pi_0^{-1}(U)$ is open.

Let $U \in \tilde{T}_0(\Sigma)$ be such that $\Pi_0^{-1}(U)$ is open. Let $x \in U$ and $y \in \Pi_0^{-1}(U)$ be such that $\Pi_0(y) = x$. There is a $B_y \in \mathcal{B}$ such that $y \in B_y \subset \Pi_0^{-1}(U)$. So $\Pi_0(B_y) \subset U$ and $x \in \Pi_0(B_y)$. Since B_y is in \mathcal{B} , $\Pi_0(B_y) \in \mathcal{F}$, so $\Pi_0(B_y)$ is open. Since x was arbitrary, U is open. This completes the proof that $\tilde{T}_0(\Sigma)$ has the quotient topology.

Finally, we show that DB acts properly discontinuously on $T_0(\Sigma^B)$. Let $x \in T_0(\Sigma^B)$. By [23, Lemma 5.2], DB acts properly discontinuously on $T(\Sigma^B)$ in its topology. Thus there is an open set $U \subset T(\Sigma^B)$ containing x such that $g(U) \cap U$ is empty for all $g \in \text{DB}$, and on which Π is one-to-one. Furthermore, $\Pi(U)$ is open in $\tilde{T}(\Sigma)$ since Π is a local homeomorphism [23]. By Theorem 4.30, $\Pi(U) \cap \tilde{T}_0(\Sigma)$ is open in $\tilde{T}_0(\Sigma)$, so there exists an $F \in \mathcal{F}$ such that $F \subset \Pi(U) \cap \tilde{T}_0(\Sigma)$ and $\Pi(x) \in F$ (note that $\Pi(x) \in \tilde{T}_0(\Sigma)$ by Proposition 5.11). So $W = (\Pi|_U)^{-1}(F)$ is in \mathcal{B} by definition, and contains x . In particular W is open, and since $W \subset U$ by construction, $g(W) \cap W$ is empty for all $g \in \text{DB}$. This completes the proof. \square

Corollary 5.15. *With the topology defined by \mathcal{B} , $T_0(\Sigma^B)$ is Hausdorff and separable.*

Proof. Let $x, y \in T_0(\Sigma^B)$, $x \neq y$. If $\Pi_0(x) \neq \Pi_0(y)$, then since $\tilde{T}_0(\Sigma)$ is Hausdorff by Theorem 4.27, there are disjoint open sets $F_x, F_y \in \mathcal{F}$ such that $\Pi_0(x) \in F_x$ and $\Pi_0(y) \in F_y$. Since \mathcal{B} is a base there are sets $B_x, B_y \in \mathcal{B}$ such that $x \in B_x$, $y \in B_y$, $\Pi_0(B_x) \subset F_x$ and $\Pi_0(B_y) \subset F_y$. Thus B_x and B_y are disjoint.

Now assume that $\Pi_0(x) = \Pi_0(y)$. Thus there is a non-trivial $[\rho] \in \text{DB}$ such that $[\rho]x = y$. Since by Theorem 5.14 DB acts properly discontinuously there is an open set V containing x such that $[\rho]V \cap V$ is empty; $[\rho]V$ is open and contains y . This completes the proof that $T_0(\Sigma^B)$ is Hausdorff.

To see that $T_0(\Sigma^B)$ is separable, let \mathfrak{A} be a countable dense subset of $\tilde{T}_0(\Sigma)$. Define $\mathfrak{B} = \{p \in T_0(\Sigma^B) : \Pi(p) \in \mathfrak{A}\}$. Since DB is countable, \mathfrak{B} is countable. To see that \mathfrak{B} is dense, observe that if U is open in $T_0(\Sigma^B)$ then, since DB acts properly discontinuously by Theorem 5.14, there is a $V \subseteq U$ on which Π is a homeomorphism onto its image. So there is a $q \in \mathfrak{A} \cap \Pi(V)$, and thus for a local inverse Π^{-1} on $\Pi(V)$ we can set $p = \Pi^{-1}(q) \in V \cap \mathfrak{B} \subseteq U \cap \mathfrak{B}$. This completes the proof. \square

Remark 5.16. It can also be shown that $T_0(\Sigma^B)$ is second countable. To see this, let \mathcal{F}' be a countable base for $\tilde{T}_0(\Sigma)$. Such a base exists by Remark 4.28. Let $\mathcal{B}' = \{B \in \mathcal{B} : \Pi_0(B) \in \mathcal{F}'\}$. It is elementary to verify that \mathcal{B}' is a base. The fact that \mathcal{B}' is countable follows from the facts that \mathcal{F}' is countable and DB is countable. Indeed, for each element F of \mathcal{F}' we can choose an element B_F of \mathcal{B}' . Each B in \mathcal{B}' is $[\rho]B_F$ for some $F \in \mathcal{F}'$ and $\rho \in \text{DB}$.

Using this base, we now define the charts on $T_0(\Sigma^B)$ that will give it a complex Hilbert space structure. For any $x \in T_0(\Sigma^B)$, let B be in the base \mathcal{B} ; therefore $F = \Pi(B)$ is in the base \mathcal{F} of $\tilde{T}_0(\Sigma)$ (see Definition 4.20). From Definition 4.22 there is the chart $G^{-1} : F \rightarrow \mathbb{C}^d \otimes (\mathcal{O}_0^{\text{qc}})^n$, where $d = 3g - 3 + n$ is the dimension of $T(\Sigma)$ and n is the number of boundary curves of Σ^B .

Definition 5.17 (Charts for $T_0(\Sigma^B)$). Given $x \in B \subset T_0(\Sigma^B)$ as above, we define the chart

$$S : B \longrightarrow \mathbb{C}^d \otimes (\mathcal{O}_0^{\text{qc}})^n$$

by $S = G^{-1} \circ \Pi_0$.

Note that to get a true chart into a Hilbert space we need to compose S with maps $\chi : \mathcal{O}_0^{\text{qc}} \rightarrow A_1^2(\mathbb{D}) \oplus \mathbb{C}$ (see (2.2) and Theorem 2.3) as in the proof of Theorem 4.30.

Theorem 5.18. *The refined Teichmüller space $T_0(\Sigma^B)$ with charts given in the above definition is a complex Hilbert manifold. With this given complex structure, Π_0 is locally biholomorphic in the sense that for every point $x \in T_0(\Sigma^B)$ there is a neighborhood U of x such that Π_0 restricted to U is a biholomorphism onto its image.*

Proof. By Corollary 5.15, we need only to show that $T_0(\Sigma^B)$ is locally homeomorphic to a Hilbert space, and exhibit an atlas of charts with holomorphic transition functions. Since Definition 5.17 defines a chart for any $x \in T_0(\Sigma^B)$, the set of such charts clearly covers $T_0(\Sigma^B)$. The maps S are clearly homeomorphisms, since G 's are biholomorphisms by Theorem 4.27 and Π_0 's are local homeomorphisms by the definition of the topology on $\tilde{T}_0(\Sigma)$.

Assume that two such charts (S, B) and (S', B') have overlapping domains. We show that $S' \circ S^{-1}$ is holomorphic on $B \cap B'$. Let $x \in B \cap B'$. Since \mathcal{B} is a base, there is a $B_1 \in \mathcal{B}$ containing x . So Π is one-to-one on B_1 ; note also that the determination of Π^{-1} on $\Pi(B_1)$ agrees with those on $\Pi(B)$ and $\Pi(B')$. So $S' \circ S^{-1} = (G')^{-1} \circ \Pi \circ \Pi^{-1} \circ G = (G')^{-1} \circ G^{-1}$ which is holomorphic by Theorem 4.27. The same proof applies to $S \circ S'^{-1}$. \square

The construction of the Hilbert manifold structure on $T_0(\Sigma^B)$ made use of an arbitrary choice of a *base rigging* $\tau \in \text{Rig}_0(\Sigma^B)$, but in fact the resulting complex structure is independent of this choice. We will show a slightly stronger result. If one considers a base Riemann

surface together with a base rigging (Σ_b^B, τ_b) to define a base point, then the *change of base point* to another such pair (Σ_a^B, τ_a) is a biholomorphism. We proceed by first examining the change of base point map for $\tilde{T}_0(\Sigma)$.

Fix two punctured Riemann surfaces Σ_a and Σ_b of the same topological type, and let $\alpha : \Sigma_a \rightarrow \Sigma_b$ be a quasiconformal map. The change of base point map α^* is defined by

$$(5.7) \quad \begin{aligned} \alpha^* : \tilde{T}_0(\Sigma_b) &\longrightarrow \tilde{T}_0(\Sigma_a) \\ [\Sigma_b, g, \Sigma_1, \phi] &\longmapsto [\Sigma_a, g \circ \alpha, \Sigma_1, \phi]. \end{aligned}$$

This is completely analogous to the usual change of base point biholomorphism for the Teichmüller space $T(\Sigma)$ (see the paragraph following Theorem 4.12). From the general definition of the Schiffer variation map in (4.3), it is worth noting that the coordinates for $\tilde{T}_0(\Sigma_0)$ as defined in (4.7) actually have this change of base point biholomorphism built in. From this observation we easily obtain the following theorem.

Theorem 5.19. *The change of base point map in (5.7) is a biholomorphism.*

Proof. The map α^* has inverse $(\alpha^*)^{-1} = (\alpha^{-1})^*$ and hence is a bijection. Consider the points $p = [\Sigma_b, g, \Sigma_1, \phi]$ and $q = \alpha^*(p) = [\Sigma_a, g \circ \alpha, \Sigma_1, \phi]$. One can choose coordinates, as in equation (4.7), for neighborhoods of p and q which use the same Schiffer variation on Σ_1 , and thus the same map ν^ϵ . In terms of these local coordinates, the map α^* is the identity map and so is certainly holomorphic. The same argument shows that $(\alpha^{-1})^*$ is holomorphic and hence α^* is biholomorphic. \square

The next task is to relate the preceding change of base point map to the one between bordered surfaces. Let Σ_b^B and Σ_a^B be bordered Riemann surfaces of type (g, n) and fix riggings $\tau_b \in \text{Rig}_0(\Sigma_b^B)$ and $\tau_a \in \text{Rig}_0(\Sigma_a^B)$. Then there exists $\rho \in \text{QC}_0(\Sigma_a^B, \Sigma_b^B)$ such that $\rho \circ \tau_a = \tau_b$. In fact one can prove a stronger statement [23, Corollary 4.7 and Lemma 4.17]: Given any quasiconformal map $\rho' : \Sigma_a^B \rightarrow \Sigma_b^B$, there exists $\rho \in \text{QC}_0(\Sigma_a^B, \Sigma_b^B)$ such that $\rho \circ \tau_a = \tau_b$ and ρ is homotopic (not rel boundary) to ρ' . The map ρ' is obtained by deforming ρ in a neighborhood of the boundary curves so as to have the required boundary values.

For such a ρ , define the change of base point map

$$(5.8) \quad \begin{aligned} \rho^* : T_0^B(\Sigma_b^B) &\longrightarrow T_0^B(\Sigma_a^B) \\ [\Sigma_b^B, f, \Sigma_1^B] &\longmapsto [\Sigma_a^B, f \circ \rho, \Sigma_1^B] \end{aligned}$$

which is just the usual change of base point map restricted to the refined Teichmüller space, together with the added condition of compatibility with the fixed base riggings.

Let Σ_b and Σ_a be the punctured surfaces obtained from Σ_b^B and Σ_a^B by sewing on caps via τ_b and τ_a respectively. Given ρ as above we have its quasiconformal extension $\tilde{\rho} : \Sigma_a \rightarrow \Sigma_b$ defined by

$$\tilde{\rho} = \begin{cases} \rho & \text{on } \overline{\Sigma_a^B} \\ \text{id} & \text{on } \mathbb{D} \end{cases}$$

as in (5.5). Let $\tilde{\rho}^*$ be the change of base point biholomorphism as in (5.7).

Lemma 5.20. *Let (Σ_b^B, τ_b) , (Σ_a^B, τ_a) , ρ , ρ^* , $\tilde{\rho}$ and $\tilde{\rho}^*$ be as above. Then the diagram*

$$\begin{array}{ccc} T_0^B(\Sigma_b^B) & \xrightarrow{\rho^*} & T_0^B(\Sigma_a^B) \\ \Pi_0 \downarrow & & \downarrow \Pi_0 \\ \tilde{T}_0(\Sigma_b) & \xrightarrow{\tilde{\rho}^*} & \tilde{T}_0(\Sigma_a) \end{array}$$

commutes.

Proof. Let $[\Sigma_b^B, f, \Sigma_1^B] \in T_0^B(\Sigma_b^B)$. We have that

$$\begin{aligned} \Pi_0 \circ \rho^*([\Sigma_b^B, f, \Sigma_1^B]) &= \Pi_0([\Sigma_a^B, f \circ \rho, \Sigma_1^B]) \\ &= [\Sigma_a^B \#_{\tau_a} \mathbb{D}, \widetilde{f \circ \rho}, \Sigma_1^B \#_{f \circ \rho \circ \tau_a} \mathbb{D}, \widetilde{f \circ \rho \circ \tau_a}] \\ &= [\Sigma_a^B \#_{\tau_a} \mathbb{D}, \widetilde{f \circ \rho}, \Sigma_1^B \#_{f \circ \tau_b} \mathbb{D}, \widetilde{f \circ \tau_b}] \\ &= [\Sigma_a, \widetilde{f \circ \rho}, \Sigma_1, \widetilde{f \circ \tau_b}] \end{aligned}$$

since $\rho \circ \tau_a = \tau_b$. On the other hand

$$\tilde{\rho}^* \circ \Pi_0([\Sigma_b^B, f, \Sigma_1^B]) = \tilde{\rho}^*([\Sigma_b^B \#_{\tau_b} \mathbb{D}, \widetilde{f}, \Sigma_1^B \#_{f \circ \tau_b} \mathbb{D}, \widetilde{f \circ \tau_b}]) = [\Sigma_b, \widetilde{f \circ \tilde{\rho}}, \Sigma_1, \widetilde{f \circ \tau_b}].$$

The claim follows from the fact that $\widetilde{f \circ \rho} = \widetilde{f} \circ \tilde{\rho}$ (Remark 5.10). \square

Theorem 5.18, Theorem 5.19, and Lemma 5.20 immediately imply the following theorem.

Theorem 5.21. *Let (Σ_b^B, τ_b) and (Σ_a^B, τ_a) be a pair of rigged bordered Riemann surfaces, with $\tau_b \in \text{Rig}_0(\Sigma_b^B)$ and $\tau_a \in \text{Rig}_0(\Sigma_a^B)$. Let $\rho \in \text{QC}_0(\Sigma_1^B, \Sigma_b^B)$ satisfy $\rho \circ \tau_a = \tau_b$. Then the change of base point map ρ^* given by equation (5.8) is a biholomorphism.*

Corollary 5.22. *The complex Hilbert manifold structure on $T_0(\Sigma^B)$ is independent of the choice of rigging $\tau \in \text{QS}_0(\Sigma^B)$.*

Proof. Apply Theorem 5.21 with $\Sigma_b^B = \Sigma_a^B = \Sigma^B$. \square

Theorem 5.23. *The inclusion map from $T_0(\Sigma^B)$ to $T(\Sigma^B)$ is holomorphic.*

Proof. Since Π has local holomorphic inverses the inclusion map from $T_0(\Sigma^B)$ to $T(\Sigma^B)$ can be locally written as $\Pi^{-1} \circ \iota \circ \Pi_0$ where $\iota : \tilde{T}_0(\Sigma) \rightarrow \tilde{T}(\Sigma)$ is inclusion. The theorem follows from the facts that Π^{-1} and Π_0 are holomorphic and ι is holomorphic by Theorem 4.30. \square

5.4. Rigged moduli space is a Hilbert manifold. In this section we show that the rigged moduli space of conformal field theory originating with Friedan and Shenker [9], with riggings chosen as in this paper, have Hilbert manifold structures.

First we define the moduli spaces. There are two models, which we will refer to as the border and the puncture model. These models are defined as follows:

Definition 5.24. Fix integers g and n , $2g - 2 + n > 0$.

(1) The border model of the refined rigged moduli space is

$$\mathcal{M}_0^B(g, n) = \{(\Sigma^B, \psi) : \Sigma^B \text{ bordered of type } (g, n), \psi \in \text{Rig}_0(\Sigma^B)\} / \sim$$

where $(\Sigma_1^B, \psi) \sim (\Sigma_2^B, \phi)$ if and only if there is a biholomorphism $\sigma : \Sigma_1^B \rightarrow \Sigma_2^B$ such that $\phi = \sigma \circ \psi$.

(2) The puncture model of the rigged moduli space is

$$\mathcal{M}_0^P(g, n) = \{(\Sigma, \psi) : \Sigma \text{ punctured of type } (g, n), \psi \in \mathcal{O}_0^{\text{qc}}(\Sigma)\} / \sim$$

where $(\Sigma_1, \psi) \sim (\Sigma_1, \phi)$ if and only if there is a biholomorphism $\sigma : \Sigma_1 \rightarrow \Sigma_2$ such that $\phi = \sigma \circ \psi$.

The puncture and border models (but with different classes of riggings) were used by [29] and [26] respectively, in the study of conformal field theory. It was understood from their inception that these rigged moduli spaces are in bijective correspondence, as can be seen by cutting and sewing caps. However, one needs to be careful about the exact classes of riggings used to make this statement precise. Replacing “bijection” with “biholomorphism” in this statement of course requires the careful construction of a complex structure on at least one of these spaces. It was shown in [23] that these two moduli spaces are quotient spaces of $T(\Sigma^B)$ by a fixed-point-free properly discontinuous group, and thus inherit a complex Banach manifold structure from $T(\Sigma^B)$. Similarly, we will demonstrate that the refined rigged moduli spaces inherit a complex Hilbert manifold structure from $T_0(\Sigma^B)$. We first need to show that the action of $\text{PModI}(\Sigma^B)$ defined by (5.1) is fixed point free and properly discontinuous.

Theorem 5.25. *The modular group $\text{PModI}(\Sigma^B)$ acts properly discontinuously and fixed-point-freely on $T_0(\Sigma^B)$. The action of each element of $\text{PModI}(\Sigma^B)$ is a biholomorphism of $T_0(\Sigma^B)$.*

Proof. Recall that $\text{DB}(\Sigma^B)$ preserves $T_0(\Sigma^B)$ by Lemma 5.4. By [23, Lemma 5.2], $\text{DB}(\Sigma^B)$ acts properly discontinuously and fixed-point freely on $T(\Sigma^B)$. Thus $\text{DB}(\Sigma^B)$ acts fixed-point freely on $T_0(\Sigma^B)$. Now let $x \in T_0(\Sigma^B)$. There is a neighborhood U of x in $T(\Sigma^B)$ such that $[\rho]U \cap U$ is empty for all $[\rho] \in \text{DB}(\Sigma^B)$. Clearly $V = U \cap T_0(\Sigma^B)$ has the same property, and is open in $T_0(\Sigma^B)$ by Theorem 5.23.

Each element $[\rho] \in \text{PModI}(\Sigma^B)$ is a biholomorphism of $T_0(\Sigma^B)$, by observing that $\rho \circ \tau = \tau$ and applying Theorem 5.21. \square

We now show that the rigged moduli spaces are Hilbert manifolds. Let Σ^B be a fixed bordered Riemann surface of type (g, n) and let $\tau \in \text{Rig}(\Sigma^B)$ be a fixed rigging. Define the mapping

$$\begin{aligned} P : T(\Sigma^B) &\longrightarrow \mathcal{M}^B(g, n) \\ [\Sigma^B, f, \Sigma_1^B] &\longmapsto (\Sigma_1^B, f \circ \tau) \end{aligned}$$

where $f \circ \tau = (f \circ \tau_1, \dots, f \circ \tau_n)$. Note that this map depends on the choice of Σ^B and τ . If we choose $\tau \in \text{Rig}_0(\Sigma^B)$, we have the map

$$P_0 = P|_{T_0(\Sigma^B)}.$$

It follows immediately from Proposition 5.8 that P_0 maps into $\mathcal{M}_0^B(g, n)$.

Theorem 5.26. *Given any $p, q \in T_0(\Sigma^B)$, $P_0(p) = P_0(q)$ if and only if $q = [\rho]p$ for some $[\rho] \in \text{PModI}(\Sigma^B)$. Moreover, P_0 is a surjection onto $\mathcal{M}_0^B(g, n)$.*

Proof. All of these claims hold in the non-refined setting by [23, Theorem 5.2]. Thus the first claim follows immediately. It was already observed that π_0 maps into $\mathcal{M}_0^B(g, n)$. To show that π_0 is surjective, observe that by [23, Theorem 5.2], for any $[\Sigma_1^B, \psi] \in \mathcal{M}_0^B(g, n)$ there is a $[\Sigma^B, f_1, \Sigma_*^B] \in T(\Sigma^B)$ such that $[\Sigma_*^B, f_1 \circ \tau] = [\Sigma_1^B, \psi]$. By composing with a biholomorphism we can assume that $\Sigma_*^B = \Sigma_1^B$ and $f_1 \circ \tau = \psi$. Thus $f_1 = \psi \circ \tau^{-1}$. Since

for $i = 1, \dots, n$ we have $\psi_i \circ \tau_i^{-1} \in \text{QS}_0(\partial_i \Sigma^B, \partial_i \Sigma_1^B)$ by Proposition 2.18, $f_1 \in \text{QC}_0(\Sigma^B, \Sigma_1^B)$. Thus $[\Sigma^B, f_1, \Sigma_1^B] \in T_0(\Sigma^B)$ and $P_0([\Sigma^B, f_1, \Sigma_1^B]) = [\Sigma_1^B, \psi]$, which completes the proof. \square

This shows that $T_0(\Sigma^B)/\text{PModI}(\Sigma^B)$ and $\mathcal{M}_0^B(g, n)$ are bijective. They are also biholomorphic.

Corollary 5.27. *The rigged moduli space $\mathcal{M}^B(g, n)$ is a Hilbert manifold and the map P_0 is holomorphic and possesses local holomorphic inverses. The Hilbert manifold structure is independent of the choice of base surface Σ^B and rigging τ .*

Proof. This follows immediately from Theorem 5.26, the fact that $\text{PModI}(\Sigma^B)$ acts fixed-point freely and properly discontinuously by biholomorphisms (Theorem 5.25), and the fact that the complex structure on $T_0(\Sigma^B)$ is independent of the choice of base rigging. \square

It was shown in [23] that the border and puncture models of the rigged moduli space are in one-to-one correspondence, and that the puncture model can be obtained as a natural quotient of $\tilde{T}_0(\Sigma)$. Those results pass immediately to the refined setting, with only very minor changes to the proofs (much as above). We will simply summarize the results here. Let Σ be a punctured Riemann surface of type (g, n) . Denote by $\text{PModP}(\Sigma)$ the modular group of quasiconformal maps $f : \Sigma \rightarrow \Sigma$ modulo the quasiconformal maps homotopic to the identity rel boundary. Elements $[\rho]$ of $\text{PModP}(\Sigma)$ act on $\tilde{T}_0(\Sigma)$ via $[\rho][\Sigma, f_1, \Sigma_1, \psi] = [\Sigma, f_1 \circ \rho, \Sigma_1, \psi]$. Define the projection map

$$\begin{aligned} Q : \tilde{T}_0(\Sigma) &\longrightarrow \mathcal{M}_0^P(g, n) \\ [\Sigma, f, \Sigma_1, \psi] &\longmapsto [\Sigma_1, \psi]. \end{aligned}$$

Finally, define the map

$$\begin{aligned} \mathcal{I} : \mathcal{M}^P(g, n) &\longrightarrow \mathcal{M}^B(g, n) \\ [\Sigma, \phi] &\longmapsto [\Sigma \setminus \overline{\phi_1(\mathbb{D}) \cup \dots \cup \phi_n(\mathbb{D})}, \phi|_{S^1}]. \end{aligned}$$

Theorem 5.28. *The moduli spaces $\mathcal{M}^P(g, n)$ and $\mathcal{M}^B(g, n)$ are in one-to-one correspondence under the bijection \mathcal{I} . Thus $\mathcal{M}^P(g, n)$ can be endowed with a unique Hilbert manifold structure so that \mathcal{I} is a biholomorphism. The map Q satisfies*

- (1) $Q(p) = Q(q)$ if and only if there is a $[\rho] \in \text{PModP}(\Sigma)$ such that $[\rho]p = [q]$
- (2) Q is surjective,
- (3) Q is holomorphic, and possesses a local holomorphic inverse in a neighborhood of every point.

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